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Chern’s conjecture for special affine manifolds. (English) Zbl 1379.57036

An affine manifold \( X \) in the sense of differential geometry is a differentiable manifold admitting an atlas of charts with values in an affine space with locally constant affine change of coordinates. Equivalently, it is a manifold whose tangent bundle admits a flat torsion free connection. Around 1955 Chern conjectured that the Euler characteristic of any compact affine manifold has to vanish. In this paper the author proves Chern’s conjecture in the case where \( X \) moreover admits a parallel volume form.


**Theorem.** An oriented \( \mathbb{R}^2 \) bundle \( E \) over a closed oriented surface \( \Sigma_g \) of genus \( g \geq 2 \) admits a flat connection \( \nabla \) if and only if \( |\chi(E)| < g \).

This strengthens Benzécri’s result: The tangent bundle of a closed connected surface \( X \) admits a flat, not necessarily torsion-free, connection if and only if \( \chi(X) = 0 \). Milnor asked whether this result generalizes to higher dimensions. J. Smillie [Comment. Math. Helv. 52, 453–455 (1977; Zbl 0357.53021)] showed that it does not. B. Kostant and D. Sullivan [Bull. Am. Math. Soc. 81, 937–938 (1975; Zbl 0313.57009)] proved Chern’s conjecture in the case where the affine structure on \( X \) is moreover complete. M. W. Hirsch and W. P. Thurston [Ann. Math. (2) 101, 369–390 (1975; Zbl 0321.57015)] proved Chern’s conjecture when the image of the holonomy homomorphism is built out of amenable groups by forming free products and taking finite extensions. Recently, M. Bucher and T. Gelander [Adv. Math. 228, No. 3, 1503–1542 (2011; Zbl 1226.53034)] proved Chern’s conjecture for varieties that are locally a product of surfaces.

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**MSC:**

57R20 Characteristic classes and numbers in differential topology
57N16 Geometric structures on manifolds of high or arbitrary dimension
57R15 Specialized structures on manifolds (spin manifolds, framed manifolds, etc.)

**Keywords:**

affine differential geometry; geometric structures on manifolds

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**References:**


[6] Cheng, Shiu Yuen; Yau, Shing-Tung, Proceedings of the 1980 Beijing (S)ymposium on (D)ifferential (G)eometry and (D)ifferential (E)quations, (V)ol. 1, 2, 3, The real {M}onge-{A}mpère equation and affine flat structures, 339-376, (1982)

[7] Cruceanu, V.; Fortuny, P.; Gadea, P. M., A survey on paracomplex geometry, Rocky Mountain J. Math., The Rocky Mountain