Hudson, Thomas; Matsumura, Tomoo
Kempf-Laksov Schubert classes for even infinitesimal cohomology theories. (English) [Zbl 1393.55005]


Editorial remark: For this item two reviews were received:

1st review by Cenap Özel (Bolu):
The purpose of this paper is to present an extension of the classical Kempf-Laksov formula for the Chow ring to an elementary family of oriented cohomology theories known as infinitesimal theories. In 1902 Giambelli described the fundamental classes of Schubert varieties of the Grassmannian in a closed, determinant expression involving Chern classes of the tautological vector bundle. Later, in 1974 Kempf and Laksov extended the Giambelli formula to Grassmannian bundles associated with a vector bundle. The authors together with T. Ikeda and H. Naruse recently generalized such computations to K-theory, obtaining a determinantal formula for the Schubert classes [Adv. Math. 320, 115–156 (2017; Zbl 1401.19008)]. This was achieved by combining the geometric input given by Kempf-Laksov resolution with an algorithmic procedure. This procedure could be further generalized to oriented cohomology and algebraic cobordism theories.

2nd review by Parameswaran Sankaran (Chennai):
The notion of a oriented cohomology theory $A^*$ in algebraic geometry was introduced by Levine and Morel, as a contravariant functor $Sm^{opp}_k \to R$ from the category of smooth quasi-projective $k$-schemes to that of graded commutative rings, satisfying several axiom, including the existence of a push-forward map $f_* : A^*(Y) \to A^{i+k}(X)$ associated to a projective morphism $f : Y \to X$ of relative dimension $d$. The formula for the tensor product of the first Chern class of line bundles yields a formal group law for $A^*$. For example, when $A^*$ is the Chow cohomology, we have the formula $c_1(L \otimes M) = c_1(L) + c_1(M)$ and the corresponding formal group law reads $F(u, v) = u + v$. There is a universal oriented cohomology, namely, the algebraic cobordism theory $\Omega^*$ over any field $k$ of characteristic 0 and Levine and Morel showed that $(Ω(Spec k), F_Ω)$ is isomorphic to $(\mathbb{L}, F_\mathbb{L})$, the Lazard ring $\mathbb{L}$ with its universal formal group law $F_\mathbb{L}$. The Lazard ring is a polynomial ring $\mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots]$ with $deg(\alpha_n) = -n$. The $n$-th infinitesimal theory $I^n_\mathbb{L}$ of the title is obtained as follows: Denote by $J_2$ the ideal of $\mathbb{L}$ consisting of elements with constant term zero. Then $J^2_\mathbb{L}$ is the ideal of decomposable elements. Let $Q := \mathbb{L}/J^2_\mathbb{L}$. The formal group law of $Q$ is given by $F_Q(u, v) = u + v + \sum_{i,j \geq 0} \binom{n}{i} (1/d_{i+j-1})\alpha_{i+j-1} u^i v^j$ where $d_i = 0$ if $i + 1$ is not a power of a prime $p$, and $d_i = 1$ otherwise. Let $Q_n = \mathbb{Z}[\alpha_n]$ where $\alpha_n^2 = 0$ and $deg(\alpha_n) = -n$. Then $I_n := Q \otimes Q_n$. The associated formal group law, denoted $\Xi_k$, is obtained as $u \otimes v = u + v + (\alpha_n/d_n) \sum_{i \leq l \leq n} \binom{n}{i} u^i v^{n+1-l}$. When $n = 2m$, it takes the form $u \otimes v = (u + v) (1 + \alpha_{2m}(\sum_{1 \leq i \leq 2m} \gamma_i (2m)^{i} u^{i} v^{2m-i}))$ where $\gamma_i (2m) = (1/d_{2m})(\binom{2m}{i} + (-1)^i)$. The fact that, in this case, $u \otimes v$ is divisible by $u + v$ leads to a simple formula for the formal inverse, namely, $\Xi_0 = -u$. This leads to a formula for the relative Segre class $S(E - F) \in I_{2m}(X)$ where $E, F$ are vector bundles over a smooth $k$-scheme $X$.

The main result of the paper gives a determinantal formula for the class of the degeneracy locus in $X$ associated to a vector bundle morphism. More precisely, let $E$ be a vector bundle of rank $d$ over $X$ with a filtration $0 = F^0 \subset \cdots \subset F^r \subset E$ and let $\lambda = \lambda_1, \ldots, \lambda_r$ be a partition of length $r$ with $\lambda_1 \leq n - d$. Then the corresponding degeneracy locus $X_\lambda$ is a subvariety of $G_d(E)$, the total space of the Grassmann bundle $G_d(E) \to X$. Denote by $S$ the tautological bundle over $G_d(E)$ of rank $d$ and by the same symbols $E, F$, the pull-back of the bundles $E, F^\lambda$ over $X$ to $G_d(E)$. The Kempf-Laksov formula expresses the Chow cohomology class $[X_\lambda]_{CH}$ as a determinant: $[X_\lambda]_{CH} = det(c_{\lambda_1+j-1}(E/F^{\lambda_j-i+i}))$. A crucial construction in its proof is a resolution of singularities of $X_\lambda$ by considering a certain flag bundle $\pi : F_{\ell_\lambda}(E) \to G_d(E)$. One has a smooth subvariety $\tilde{X}_\lambda^{KL} \subset F_{\ell_\lambda}(E)$ such that $\pi$ restricts to a surjective birational morphism $\tilde{X}_\lambda^{KL} \to X_\lambda$ and is referred to as the Kempf-Laksov resolution. Denoting the fundamental class of $\tilde{X}_\lambda^{KL}$ as $[\tilde{X}_\lambda^{KL}]$, we have $[\tilde{X}_\lambda^{KL}] = \pi_* [F_{\ell_\lambda}(E)]$.
by $[\tilde{X} \to F\ell(E)] \in I_{2m}^*(F\ell(E))$, the authors obtain, in Theorem 4.6, the following formula for the ‘Kempf-Laksov’ class $\kappa_\lambda := \pi_*([\tilde{X}^{KL}_{\lambda} \to F\ell(E)]) \in I_{2m}(G_d(E))$:

$$\kappa_\lambda = \det A + \alpha_{2m} \sum_{-m+1 \leq i \leq m-1} (-1)^{m+l+1} \sum_{1 \leq a < b \leq r} \det A_{l,a,b}$$

where $A = (A_{k+i,j})_{1 \leq i,j \leq r}$ with $k_i := \lambda_i - i + d$, $A_{j,i} := S_j((S^\vee - (E/F)^i)^\vee)$, the matrix $A_{l,a,b}$ is obtained from $A$ by replacing the $a^{th}$ and the $b^{th}$ row by $(A_{k+i,j}^{\lambda+a})_{1 \leq j \leq r}$ and $(A_{k+i,j}^{\lambda+b})_{1 \leq j \leq r}$ respectively. Similar formula for the class of degeneracy loci in $I_{2m}^*(LG(E))$ in the case of the Lagrangian Grassmann bundle $LG(E)$ of a symplectic vector bundle $E$ involving the so-called multi-Schur Pfaffians have also been obtained.

For the entire collection see [Zbl 1382.14002].

Reviewer: Cenap Özel (Bolu); Parameswaran Sankaran (Chennai)

**MSC:**

- 55N20 Generalized (extraordinary) homology and cohomology theories in algebraic topology [Cited in 3 Documents]
- 55N22 Bordism and cobordism theories and formal group laws in algebraic topology
- 14N15 Classical problems, Schubert calculus
- 14M15 Grassmannians, Schubert varieties, flag manifolds

**Keywords:**

oriented cohomology theories; algebraic cobordism; Schubert calculus; Lazard ring; infinitesimal theories; Kempf-Laksov formula; Giambelli formula

**Full Text:** arXiv