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Restricted Stirling and Lah number matrices and their inverses. (English) [Zbl 1400.05029]


Summary: Given $R \subseteq N$ let $\binom{n}{k}_R$, $\binom{n}{k}_R^*$, and $L(n,k)_R$ count the number of ways of partitioning the set $[n] := \{1,2,\ldots,n\}$ into $k$ non-empty subsets, cycles and lists, respectively, with each block having cardinality in $R$. We refer to these as the $R$-restricted Stirling numbers of the second kind, $R$-restricted unsigned Stirling numbers of the first kind, and Lah numbers, respectively. Note that the classical Stirling numbers of the second kind, unsigned Stirling numbers of the first kind, and Lah numbers are $\binom{n}{k} = \binom{n}{k}_N, \binom{n}{k} = \binom{n}{k}_N$ and $L(n,k) = L(n,k)_N$, respectively. It is well-known that the infinite matrices $[\binom{n}{k}]_{n,k \geq 1}$ and $[L(n,k)]_{n,k \geq 1}$ have inverses $[(-1)^{n-k}\binom{n}{k}]_{n,k \geq 1}$ and $[(-1)^{n-k}L(n,k)]_{n,k \geq 1}$ respectively. The inverse matrices $[\binom{n}{k}_R]_{n,k \geq 1}$, $[\binom{n}{k}_R^*]_{n,k \geq 1}$ and $[L(n,k)_R]_{n,k \geq 1}$ exist and have integer entries if and only if $1 \in R$. We express each entry of each of these matrices as the difference between the cardinalities of two explicitly defined families of labeled forests. In particular the entries of $[\binom{n}{k}_R]^{-1}_{n,k \geq 1}$ have combinatorial interpretations, affirmatively answering a question of J. Y. Choi et al. [J. Comb. Theory, Ser. A 113, No. 6, 1050–1060 (2006; Zbl 1151.05003)]. If we have $1,2 \in R$ and if for all $n \in R$ with $n$ odd and $n \geq 3$, we have $n \pm 1 \in R$, we additionally show that each entry of $[\binom{n}{k}_R]^{-1}_{n,k \geq 1}$, $[\binom{n}{k}_R^*]_{n,k \geq 1}$ and $[L(n,k)_R]^{-1}_{n,k \geq 1}$ is up to an explicit sign the cardinality of a single explicitly defined family of labeled forests. With $R$ as before we also do the same for restriction sets of the form $R(d) = \{dr - \frac{r(r-1)}{2}: r \in R\}$ for all $d \geq 1$. Our results also provide combinatorial interpretations of the $k$th Whitney numbers of the first and second kinds of $\Pi_n^d$, the poset of partitions of $[n]$ that have each part size congruent to 1 mod $d$.

MSC:

05A18 Partitions of sets
05A15 Exact enumeration problems, generating functions
11B73 Bell and Stirling numbers
11B75 Other combinatorial number theory

Keywords:

Stirling numbers; Lah numbers; Riordan matrix; Riordan group; reversion; Lagrange inversion; Whitney numbers; restricted partition poset

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References:


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