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Lower complexity bounds for positive contactomorphisms. (English) [Zbl 1406.53080]

Let \( Q \) be a manifold of dimension at least \( n \geq 2 \), and let \( S^* Q \) be its spherisation (the projectivisation of its cotangent bundle). The \((2n-1)\)-dimensional manifold \( S^* Q \) has a natural co-oriented contact structure \( \xi \). A contactomorphism \( \varphi \) of \( S^* Q \) is positive if there is a path of contactomorphisms, from the identity to \( \varphi \), whose time derivative is always positively transverse to \( \xi \), or equivalently, if \( \varphi \) is the result of a time-dependent Reeb flow.

The main results of this paper say, roughly, that if \( Q \) is sufficiently topologically complex, then any positive contactomorphism \( \varphi \) of \( S^* Q \) has fast volume growth. The requisite topological complexity is measured by the growth of the fundamental group of \( Q \), or by the growth of the homology of the space of contractible loops in \( Q \). The volume growth of \( \varphi \), and the topological complexity of \( Q \), can both be measured in exponential or polynomial terms. The main result consists of a statement about exponential growth, and a statement about polynomial growth.

The precise statements are as follows; precise definitions of growth rates are given below. Let \( Q \) be closed of dimension at least 2, let \( \varphi \) be a positive contactomorphism of \( S^* Q \), and let \( q \in Q \). Firstly (regarding exponential growth), if \( \gamma_{\exp}(\pi_1(Q)) > 0 \), or if \( \pi_1(Q) \) is finite and \( \gamma_{\exp} (\Omega Q_0(q)) > 0 \), then

\[
\gamma_{\text{vol}, \exp}(\varphi) \geq \gamma_{\text{vol}, \exp}(\varphi; S^*_Q Q) > 0,
\]

where \( S^*_Q Q \subset S^* Q \) is the fibre of \( S^* Q \) over \( q \in Q \). Secondly (regarding polynomial growth), if \( \gamma_{\text{pol}}(\pi_1(Q)) \) and \( \gamma_{\text{pol}}(\Omega Q_0(q)) \) are finite, then

\[
\gamma_{\text{vol}, \text{pol}}(\varphi) \geq \gamma_{\text{vol}, \text{pol}}(\varphi; S^*_Q Q) \geq \gamma_{\text{pol}}(\pi_1(Q)) + \gamma_{\text{pol}}(\Omega Q_0(q)) - 1.
\]

The measures of topological complexity in this statement are defined as follows. Taking \( q \in Q \), denote by \( \Omega Q_0(q) \) the connected component of the loop space based at \( q \) consisting of contractible loops. The exponential and polynomial growth of homology of \( \Omega Q_0(q) \) are given by

\[
\gamma_{\text{exp}}(\Omega Q_0(q)) = \sup_{p \in \mathbb{P}} \liminf_{m \to \infty} \frac{1}{m} \log m \sum_{k=0}^m \dim (H_k(\Omega Q_0(q); F_p)),
\]

\[
\gamma_{\text{pol}}(\Omega Q_0(q)) = \sup_{p \in \mathbb{P}} \liminf_{m \to \infty} \frac{1}{\log m} \log m \sum_{k=0}^m \dim (H_k(\Omega Q_0(q); F_p)),
\]

where \( \mathbb{P} \) is the set of primes, together with zero, and \( F_p \) is the field \( \mathbb{Z}/p\mathbb{Z} \), with \( \mathbb{F}_0 = \mathbb{Q} \). Similarly, \( \gamma_{\text{exp}}(\pi_1(Q)) \) and \( \gamma_{\text{pol}}(\pi_1(Q)) \) denote the exponential and polynomial growth of the fundamental group \( \pi_1(Q) \) with respect to some set of generators; any set of generators gives the same result.

On the other hand, the exponential and polynomial volume growth rates of a diffeomorphism \( \varphi \) of a Riemannian manifold \( M \) (the result is independent of the choice of metric) are respectively given by

\[
\gamma_{\text{vol}, \text{pol}}(\varphi) = \sup_{S} \liminf_{m \to \infty} \frac{1}{m} \log \text{Vol}(\varphi^m(S)),
\]

\[
\gamma_{\text{vol}, \exp}(\varphi) = \sup_{S} \liminf_{m \to \infty} \frac{1}{m} \log \text{Vol}(\varphi^m(S)),
\]

where the suprema are over all compact submanifolds \( S \) of \( M \).

Work of Y. Yomdin [Isr. J. Math. 57, 285-300 (1987; Zbl 0641.54036)] and S. E. Newhouse [Ergodic Theory Dyn. Syst. 8, 283-299 (1988; Zbl 0638.58016)] shows that when \( \varphi \) is \( C^\infty \) smooth, the topological entropy \( h_{\exp}(\varphi) \) of \( \varphi \) is equal to \( \gamma_{\text{vol}, \exp}(\varphi) \). So the author obtains the corollary that when the hypotheses
of the exponential result hold, \( \varphi \) has positive topological entropy.


The setup is as follows. The spherisation \( S^*Q \) is realised as a cosphere bundle in \( T^*Q \). A positive contactomorphism \( \varphi \) of \( S^*Q \) arises from a time-dependent family \( \varphi^t \) with \( \varphi^0 = 1 \) and \( \varphi^1 = \varphi \); this family can be chosen so as to be generated by a suitable Hamiltonian, and extended to \( T^*Q \setminus Q \) (the complement of the zero section). A result of Frauenfelder-Labrousse-Schlenk then states that

\[
\gamma_{\text{vol,exp}}(\varphi^t; S^*_q Q) \geq \gamma_{\text{vol,exp}}(\varphi^t; \hat{D}_q^* Q) \quad \text{and} \quad \gamma_{\text{vol,exp}}(\varphi^t; \hat{D}_q^* Q) \geq \gamma_{\text{vol,exp}}(\varphi^t; \hat{D}_q^* Q) - 1,
\]

where \( \hat{D}_q^* Q \) is the punctured sublevel disc of the sphere \( S^*_q Q \).

The Albers-Frauenfelder version of Rabinowitz Floer homology considers an action functional on \( W^{1,2} \) paths between two fibres \( T^*_q Q \) and \( T^*_q Q' \). Given a path of contactomorphisms \( \varphi^t \), the author constructs a Hamiltonian \( H^t \) and hence obtains an appropriate action functional. The Rabinowitz Floer chain complex \( RFC^T(\varphi^t; q, q') \) is generated by critical points of the action functional with action \( \leq T \), which are closely related to Hamiltonian orbits of \( H^t \) and hence to orbits of \( \varphi^t \). Its boundary operator counts solutions of a negative gradient flow with respect to a suitable \( L^2 \) metric. An action filtration is given by inclusions \( \iota^t: RFC^T \to RFC^{T'} \). The positive part of \( RFC^+ \) is given by \( RFC^T_+ = RFC^T / \partial RFC^0 \), and its homology is the Rabinowitz Floer homology \( RFH^T_+ (\varphi^t; q, q') \).

The author proceeds to relate volume growth of \( \hat{D}_q^* Q \) under \( \varphi^T \) to orbits of \( \varphi^t \), and for generic \( q' \), \( \varphi^T(\hat{D}_q^* Q) \) intersects \( \hat{D}_{q'}^* Q \) transversely and hence, for a generic set \( Q_{gen} \subset Q \) we have

\[
\text{Vol}(\varphi^T(\hat{D}_q^* Q)) \geq \int_{Q_{gen}} \# (\varphi^T(\hat{D}_q^* Q) \cap D_{q'}^* Q) \ dq'.
\]

The points of \( \varphi^T(\hat{D}_q^* Q) \cap D_{q'}^* Q \) correspond to certain orbits of \( \varphi^t \) from \( S^*_q Q \) to \( S^*_q Q' \), which are counted by the Rabinowitz Floer chain complex \( RFC^T_+ (\varphi^t; T^*_q Q, T^*_q Q') \). The dimension of homology is even smaller, so for a slightly smaller generic set \( Q_{gen} \) of \( q' \),

\[
\text{Vol}(\varphi^T(\hat{D}_q^* Q)) \geq \int_{Q_{gen}} \dim(\iota^T_{+} \varphi^T; T^*_q Q, T^*_q Q') \ dq'.
\]

The key ingredient in the proof is a result stated by Albers-Frauenfelder (proved in Section 4 of the present paper), that positive exponential growth of RFH is preserved under a deformation of the flow \( \varphi^t \). Moreover, the polynomial growth rate is preserved under such deformations. Thus, \( \varphi^t \) may be deformed to a geodesic flow \( \varphi^g \), and the polynomial growth, and positivity of exponential growth, are preserved.

A theorem of W. J. Merry [Geom. Dedicata 171, 345–386 (2014; Zbl 1312.53111)] is then used, which shows that the Rabinowitz Floer homology of a geodesic flow is isomorphic to the Morse homology of the energy functional on the space of paths in \( Q \) from \( q \) to \( q' \). Finally, work of Gromov, Paternain, and Frauenfelder-Schlenk is applied, which relates the growth of this Morse homology to the topological complexity of \( Q \).

Two examples are given to show that, without an assumption of positivity, the topological entropy of a contactomorphism may be zero or positive – thus limiting potential generalisations of the main result. Both examples are of closed manifolds \( Q \), with \( \pi_1(Q) \) having exponential growth, and non-negative contactomorphisms \( \varphi \) on \( S^*Q \). The first example, which involves the quotient of a semidirect product \( \mathbb{R}^2 \ltimes \mathbb{R} \) by a cocompact lattice of exponential growth, has \( h_{\text{top}}(\varphi) = 0 \). The second example, which involves a geodesic flow on the unit cotangent bundle of a closed orientable surface of genus \( \geq 2 \) with a suitably chosen metric, has \( h_{\text{top}}(\varphi) > 0 \).

Reviewer: Daniel Mathews (Monash)
MSC:

53D10  Contact manifolds (general theory)
53D35  Global theory of symplectic and contact manifolds
53D40  Symplectic aspects of Floer homology and cohomology

Cited in 3 Documents

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References:


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