Higham, Nicholas J.; Mary, Theo

The aim of the paper consists in the convergence acceleration of iterative methods for the solution of large ill-conditioned linear algebraic equations \( Ax = b, A \in \mathbb{R}^{n \times n} \) nonsingular.

A classical approach to accelerate the convergence is based on a computed LU factorization of \( A \) with low accuracy, and to solve the preconditioned system \( MAx = Mb, M = \hat{U}^{-1}\hat{L}^{-1} \). But, the convergence may be slow in the case of ill-conditioned systems. That is demonstrated for (i) the low floating point precision LU factorization by E. Carson and N. J. Higham [SIAM J. Sci. Comput. 40, No. 2, A817–A847 (2018; Zbl 1453.65067)], using LU factors in single precision, working precision in double precision, and residuals in quadruple precision, (ii) for the incomplete LU factorization [Y. Saad, Iterative methods for sparse linear systems. 2nd ed. Philadelphia, PA: SIAM Society for Industrial and Applied Mathematics (2003; Zbl 1031.65046)], and for the block low-rank LU factorization [P. Amestoy et al., SIAM J. Sci. Comput. 39, No. 4, A1710–A1740 (2017; Zbl 1372.65089)].

To overcome the disadvantages, a new preconditioner is derived.

The main idea is based on the observation that real-life ill-conditioned matrices have a small number of small singular values. The corresponding inverse matrix has then a small number of large singular values, that is, it is numerically low-rank.

Based on the expectation that the error matrix \( E = MA - I \) could be also numerically low-rank, the authors derive corresponding sufficient conditions for it. Exploiting then the low-rank property of \( E \), a novel preconditioner is built using \( M = (I + E_k)^{-1}\hat{U}^{-1}\hat{L}^{-1} \) rather than \( M, E_k \) is a rank-\( k \) approximation of the error vector \( E \).

To compute an efficient low-rank approximation \( E_k \), the authors derive a matrix-free approach. Only matrix-vector products have to be performed. In the two randomized sampling algorithms the \( E_k \) are computed as a truncated singular value decomposition of \( E \) [N. Halko et al., SIAM Rev. 53, No. 2, 217–288 (2011; Zbl 1269.65043)], [E. Liberty et al., Proc. Natl. Acad. Sci. USA 104, No. 51, 20167–20172 (2007; Zbl 1215.65080)].

The authors mention that in the case \( k \ll n \) the solution with \( (I + E_k) \) can be particularly efficiently done using the Sherman-Morrison-Woodbury formula.

The convergence improvements achieved by the new preconditioner are analyzed using GMRES-based iterative refinements.

The numerical rank of the error matrix is theoretically and experimentally investigated. A large set of both random real-life sparse and dense matrices, and matrices from the SuiteSparse Matrix Collection are applied in order to verify the advantages of the new preconditioner. The cost and accuracy of the preconditioner are influenced by a number of parameters, which are discussed in detail. Results are demonstrated by a number of tables and figures for the moderate large systems.

Reviewer: Georg Hebermehl (Berlin)

MSC:

65F10 Iterative numerical methods for linear systems
65F08 Preconditioners for iterative methods
65F05 Direct numerical methods for linear systems and matrix inversion
65F25 Orthogonalization in numerical linear algebra
65F30 Other matrix algorithms (MSC2010)
68Q87 Probability in computer science (algorithm analysis, random structures, phase transitions, etc.)
Keywords:
ill-conditioned linear system; matrix factorization; preconditioning; low-rank approximations; mixed precision iterative refinement; incomplete LU factorization; block low-rank LU factorization; randomized sampling algorithm; GMRES

Software:
SparseMatrix; SuiteSparse; advanpix

Full Text: DOI

References:

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