A fundamental optimization problem in statistical inference is the maximum likelihood estimation for discrete data. Algebraic geometry enters the picture since this maximum likelihood estimation can be formulated in terms of the solutions of a system of polynomial equations. Briefly, given a projective variety $V \subseteq \mathbb{P}^{n-1}$ over the complex field $\mathbb{C}$ with homogeneous coordinates $p = [p_1, \ldots, p_n]$, by considering points $p = [p_1, \ldots, p_n] \in V$ with $p_i \geq 0$ as a family of probability distributions of a discrete random variable, the likelihood function is given by

$$\ell_u(p) = \frac{p_1^{u_1} \cdots p_n^{u_n}}{(p_1 + \cdots + p_n)^{u_1 + \cdots + u_n}}$$

for a data vector $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$. The problem is to find a probability distribution $\hat{p} \in V$ which maximizes $\ell_u$. Such a $\hat{p}$ is called a maximum likelihood estimate and the idea is to identify it by computing all critical points on $V$ of the function $\ell_u$. The maximum likelihood degree of the algebraic statistical model, denoted $\text{mldeg}(V)$, introduced by F. Catanese et al. [Am. J. Math. 128, No. 3, 671–697 (2006; Zbl 1123.13019)] and S. Hosten et al. [Found. Comput. Math. 5, No. 4, 389–407 (2005; Zbl 1097.13035)], is the number of critical points of the likelihood function $\ell_u$ on $V_{\text{reg}} \setminus H$ for a generic vector $u$, where $V_{\text{reg}}$ is the regular locus of $V$ and $H$ is the union of the coordinate planes and the hyperplane defined by $p_1 + \cdots + p_n = 0$.

By focusing on discrete exponential models in algebraic statistics, given by monomial parametrizations, in the algebraic geometry side the corresponding objects are toric varieties and in this paper the authors study the maximum likelihood degree of scaled toric varieties. Here, a toric variety $V^c \subseteq \mathbb{P}^{n-1}$ scaled by $c = (c_1, \ldots, c_n) \in (\mathbb{C}^*)^n$ is the affine cone given by the closure in $\mathbb{C}^n$ of the image of the monomial parametrization $\psi^c : (\mathbb{C}^*)^d \to (\mathbb{C}^*)^n$ defined by

$$\psi^c(s, \theta_1, \ldots, \theta_d-1) = (c_1 s \theta_1^{a_1}, c_2 s \theta_2^{a_2}, \ldots, c_n s \theta_n^{a_n})$$

where $\theta = (\theta_1, \ldots, \theta_{d-1}) \in (\mathbb{C}^*)^{d-1}$ and $a_i \in \mathbb{Z}^{d-1}$ are the columns of a full rank $(d-1) \times n$ integer matrix $A$. For $c = (1, \ldots, 1)$ the variety $V^{(1,\ldots,1)}$ is the usual toric variety $V$ corresponding to the lattice polytope $Q = \text{conv}(A)$ defined by the complex hull of the lattice points $a_i$.

The main result of the paper, Theorem 13, shows that for a fixed $c \in (\mathbb{C}^*)^n$, the maximum likelihood degree of the scaled toric variety $V^c$ satisfies that $\text{mldeg}(V^c) < \deg(V^{(1,\ldots,1)})$ if and only if $c$ is in the zero locus of the principal $A$-determinant of the toric variety $V^{(1,\ldots,1)}$, as defined in Theorem 1.2, Chapter 10 of [J. M. Gelfand et al., Discriminants, resultants, and multidimensional determinants. Boston, MA: Birkhäuser (1994; Zbl 0827.14036)].

The paper includes the computation of the maximum likelihood degree of rational normal scrolls and some Veronese-type varieties. It also investigates properties of this degree for some scaled Segre embeddings, hierarchical log-linear models and graphical models. The last section of the paper uses homotopy continuation to track maximum likelihood degree estimates between different scalings of a given algebraic statistical model.

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MSC:
14M25 Toric varieties, Newton polyhedra, Okounkov bodies
14Q15 Computational aspects of higher-dimensional varieties
62F10 Point estimation
68W30 Symbolic computation and algebraic computation
13P25 Applications of commutative algebra (e.g., to statistics, control theory, optimization, etc.)

Keywords:
toric varieties; computational algebraic geometry; point estimation; symbolic computation

Software:
PHCpack; Macaulay2

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