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Resultants and the Borcherds Φ-function. (English) Zbl 1425.14033

The Borcherds Φ function referred to in the title is what is sometimes called the Borcherds-Enriques form, a weight 4 automorphic form on the period domain of Enriques surfaces. In the relatively short time since its discovery or invention [R. E. Borcherds, Topology 35, No. 3, 699–710 (1996; Zbl 0886.14015)] it has found several applications in algebraic geometry, but an algebraic construction of it has proved elusive. This paper largely makes good that deficiency.

The Borcherds-Enriques form is loosely analogous to the Dedekind η-function and the approach here is to look for a counterpart to the formula for the value of η at the period point τα,β of an elliptic curve

\[ E = \{y^2 = 4x^3 - g_2x - g_3\} \]

together with a basis \{α, β\} of \(H_1(E, \mathbb{Z})\):

\[ \eta(\tau_{α, β})^{24} = (g_2^3 - 27g_3^2) \left( \frac{1}{2\pi} \int_{α} \frac{dx}{y} \right). \]

For an Enriques surface \(Y\) the rigidification needed to obtain an unambiguous period (corresponding to the choice of basis above) is a marking of the covering \(K3\) surface, but we can avoid making this choice if, at first, we consider only the Petersson norm \(\|Φ(Y)\|\). To give an expression for this, the authors first give a concrete representation of \(Y = Y_{(f,g,h)}\) as the quotient of \(X_{(f,g,h)} \subset \mathbb{P}^5\) by the involution \(\iota\) of \(\mathbb{P}^5\) that multiplies the last three homogeneous coordinates by \(-1\). Here \(f\), \(g\) and \(h\) are carefully chosen quadrics, namely each is the sum \(f = f_1 + f_2\) (etc.) of a quadratic form in \(x_1, x_2, x_3\) and one in \(x_4, x_5, x_6\). It is known that every Enriques surface arises this way.

This immediately yields a standard, completely explicit, 2-form \(ω\) on \(X_{f,g,h} \subset \mathbb{P}^5\) and the first main theorem says that

\[ \|Φ(Y_{(f,g,h)})\|^2 = |R(f_1, g_1, h_1)R(f_2, g_2, h_2)| \left( \frac{2}{π} \int_{X_{(f,g,h)}} ω ∧ \varpi \right)^4. \]

where \(R\) denotes the resultant.

From this the authors deduce a similar formula for \(Φ\) itself, in a form depending on a choice of cusp (essentially just a choice of level 1 or level 2).

Some interesting consequences follow. One of them is a Thomae-type result expressing \(\|Φ\|\) in certain cases as the fourth power of the Petersson norm of a certain theta Fourier series, similarly to the results of K. Matsumoto and T. Terasoma [J. Reine Angew. Math. 669, 121–149 (2012; Zbl 1251.14028)]. (In fact, as they point out, \(Φ\) itself is described by theta series in \([E. Freitag and R. Salvati Manni, Asian J. Math. 21, No. 3, 483–498 (2017; Zbl 1441.11090)]) From this they deduce an expression for any genus 2 even theta constant as a Borcherds product, refining a result of V. A. Grötschen and V. V. Nikulin [Am. J. Math. 119, No. 1, 181–224 (1997; Zbl 0914.11020)] for the product of all such theta constants.

The proof of the formula for the Petersson norm is based on a comparison of \(\partial \bar{∂} \log\) of the two sides of the equality. These are interpreted as currents on a certain Grassmannian: the necessary computation is straightforward where the resultants do not vanish, but there it requires a careful and ingenious examination of the singularity of \(\|Φ\|\). This is the main new idea in the paper: the rest of the proofs, though not in the least routine, and technically hard, are along the lines that an expert might expect.

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