Stibitz, Charlie; Zhuang, Ziquan

K-stability of birationally superrigid Fano varieties. (English) [Zbl 1425.14035]

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Let $X$ be a $\mathbb{Q}$-Fano variety (i.e. $X$ has klt singularities and the $\mathbb{Q}$-divisor $-K_X$ is ample) over $\mathbb{C}$ with Picard number one. Recall that the alpha invariant $\alpha(X)$ of $X$ is the supremum of all $t > 0$ such that the pair $(X, tD)$ is log canonical for every $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$. Suppose also that $X$ is birationally superrigid (or, more generally, $K_X$ is lc for every movable boundary $M \sim_{\mathbb{Q}} -K_X$). In the paper under review, the authors prove (see Theorem 1.2 in the text) that if $\alpha(X) \geq \frac{1}{2}$ (resp. $\alpha(X) > \frac{1}{2}$), then $X$ is $K$-semistable (resp. $K$-stable). This generalizes in particular the result of K. Fujita [J. Inst. Math. Jussieu 18, No. 3, 519–530 (2019; Zbl 1409.14008)] that a smooth hypersurface $X \subset \mathbb{P}^n$ of degree $n \geq 4$ is K-stable. Furthermore, the authors also prove (Theorem 1.6) that if $K_X$ generates the class group of $X$ and the linear system $| -K_X|$ is free, then $\alpha(X) \geq 1/(n + 1)$. Some other applications can be traced in the text.

Let $F$ be a simple divisorial valuation on $X$. One may identify $F$ with a prime divisor on a birational model $\pi : Y \longrightarrow X$. Recall that $F$ is called dreamy if the graded algebra

$$\bigoplus_{k,j \in \mathbb{Z}_{\geq 0}} H^0(Y, -k r \pi^* K_X - j F)$$

is finitely generated for some $r$ such that $rK_X$ is Cartier. Then $X$ is $K$-semistable (resp. $K$-stable) iff

$$\beta(F) := A_X(F) \cdot ((-K_X)^n) - \int_0^\infty \text{vol}_X(-K_X - xF)dx \geq 0$$

(resp. $> 0$) for all simple $F$ (see K. Fujita [J. Reine Angew. Math. 751, 309–338 (2019; Zbl 1435.14039]) and C. Li [Duke Math. J. 166, No. 16, 3147–3218 (2017; Zbl 1409.14008)])]. Here $A_X(F)$ is the log discrepancy of $F$ and $\text{vol}_X(-K_X - xF) := \text{vol}_Y(-\pi^* K_X -xF)$.

The proof proceeds by assuming that $\beta(F) < 0$ for some $F$ as above. This is then brought to contradiction with $\alpha(X) \geq \frac{1}{2}$ similarly as in [K. Fujita, Kyoto J. Math. 59, No. 2, 399–418 (2019; Zbl 1419.14065)]. The claim about $\alpha(X) \geq 1/(n + 1)$ is proved by showing that $\text{lt}(X, D) \geq 1/(n + 1)$ for every irreducible $D \sim_{\mathbb{Q}} -K_X$. Namely, the pair $(X, D)$ is lc in codimension 1 because $K_X$ generates the class group, which implies that the multiplier ideal $\mathcal{J}(X, (1 - \epsilon)D)$, $0 < \epsilon \ll 1$, defines a subscheme of codimension $\geq 2$. Then by Nadel’s vanishing, $H^i(X, \mathcal{J}(X, (1 - \epsilon)D) \otimes \mathcal{O}_X(-rK_X)) = 0$, all $i > 0$, $r \geq 0$, which yields (by Castelnuovo-Mumford regularity) the movable linear system

$$\mathcal{M} := \mathcal{J}(X, (1 - \epsilon)D) \otimes \mathcal{O}_X(-nK_X)].$$

Now, assuming that $\text{lt}(X, D) < 1/(n + 1)$ for a given $D$, one easily gets that $(X, M)$ is not lc for the movable boundary $M = (1/n)\mathcal{M}$, a contradiction.

Reviewer: Ilya Karzhemanov (Moscow)

MSC:

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32Q20 Kähler-Einstein manifolds

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