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Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell-Weil rank. (English) Zbl 1428.11122

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Given a curve C defined over a field K, let $\#C(K)$ denote the number of its K-rational points. By *G. Faltings*' result [in: Barsotti symposium in algebraic geometry. Memorial meeting in honor of Iacopo Barsotti, in Abano Terme, Italy, June 24-27, 1991. San Diego, CA: Academic Press. 175–182 (1994; Zbl 0823.14009)], $\#C(K)$ is finite if its genus is ≥ 2 and K is a number field, and given a number field K and an integer $g \geq 2$, the set of $\#C(K)$ where C varies over the curves of genus g defined over K is of great interest in number theory. The paper under review introduces a nice survey on finiteness conjectures on the sets that are similar to the aforementioned one, and particularly focuses on the following version of Mazur: Given integers $d \geq 1$, $g \geq 2$, and $r \geq 0$, consider the set of $\#C(K)$ where K varies over the number fields of degree d, and C varies over the curves of genus g defined over K such that the Mordell-Weil rank of its Jacobian is r. Then, the aforementioned set is finite.

The author of the paper under review proves this conjecture of Mazur for integers $g \geq 3$, $d \geq 1$, $r \geq 0$ such that $r \leq g - 3$, and hyperelliptic curves C/K. The author's proof is based on the upper bound available from Chabauty's method, and the difficulty is that the size of the special fiber which is a term in Chabauty's upper bound is unbounded as C varies. The author overcomes this difficulty by imposing additional linear conditions on the differential entry of the Chabauty-Coleman pairing, and this results in the cost of requiring $r \leq g - 3$. The author's idea does in fact generalize to arbitrary curves, and it is demonstrated in [*E. Katz* et al., Duke Math. J. 165, No. 16, 3189–3240 (2016; Zbl 1428.11121)].

Reviewer: Sungkon Chang (Savannah)

MSC:

11G30 Curves of arbitrary genus or genus $\neq 1$ over global fields
14G05 Rational points
14G25 Global ground fields in algebraic geometry
14H25 Arithmetic ground fields for curves
14H40 Jacobians, Prym varieties

Keywords:

rational points on curves; uniform bounds; Chabauty's method; p-adic integration; Mordell-lang conjecture; Zilber-Pink conjectures

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