Morrow, Matthew

A variational Tate conjecture in crystalline cohomology. (English) Zbl 1430.14026

J. Eur. Math. Soc. (JEMS) 21, No. 11, 3467-3511 (2019).

The article under review proves a crystalline version of the variational Tate conjecture for divisors. Here is the precisely statement. Let $f : X \to S$ be a smooth projective morphism of smooth varieties over a perfect field of characteristic $p > 0$, let $s \in S$ be a closed point. Let $c \in \mathrm{H}^2_{\mathrm{cris}}(X/W(k))[1/p]$ be the cycle class of an element in $\mathrm{CH}^1(X_s) \otimes \mathbb{Q}$. Then c falls in the image of the restriction map $\mathrm{H}^2_{\mathrm{cris}}(X/W(k))[1/p] \to \mathrm{H}^2_{\mathrm{cris}}(X_s/W(k))[1/p]$ (c is "horizontal") if and only if there exists $z \in \mathrm{CH}^1(X) \otimes \mathbb{Q}$ such that $\mathrm{cl}(z_{X_s})| = c$.

The theorem is a consequence of its local analogue (when S is the spectrum of $A = k[\![t_1, \dots, t_m]\!]$), discussed in Section 3. We give a rough sketch of the proof of this local theorem. Set $Y_r = X \otimes A/(t)^{r+1}$, $Y = Y_0$.

(a) The trace map from the K-theory spectrum to the topological cyclic homology spectrum induces a homotopy pullback

$$\operatorname{holim} K(Y_i) = TC(X;p) \times^h_{TC(Y;p),\mathrm{tr}} K(Y).$$

which induces a ladder of exact sequences of homotopy groups. This is due to [*B. Dundas* and *M. Morrow*, Ann. Sci. École Norm. Sup. (4) 50, 201-238 (2017; Zbl 1372.19002)]. The $\pi_0 \otimes \mathbb{Q}$ of the topological cyclic homology is the sum of $\mathrm{H}^i(X, W\Omega^i_{X,\log})[1/p]$ (the continuous cohomology of logarithmic de Rham-Witt sheaves, see [*A. Shiho*, J. Math. Sci., Tokyo 14, No. 4, 567–635 (2007; Zbl 1149.14013)]).

(b) Spelling out the meaning of the exactness at π_0, one concludes that a class $z \in K_0(Y)$ lifts to $\lim_r K_0(Y_r)[1/p]$ if and only if $\mathrm{ch}^{\log}(z) \in \bigoplus \mathrm{H}^i(Y, W\Omega^i_{Y,\log})[1/p]$ lifts to a class in $\bigoplus \mathrm{H}^i(X, W\Omega^i_{X,\log})[1/p]$

(c) One shows, by studying the canonical map $\varepsilon_{\mathbb{Q}} : \mathrm{H}^i(X, W\Omega^i_{X,\log})[1/p] \to \mathrm{H}^{2i}_{\mathrm{cris}}(X/W(k))[1/p]$, that the desired crystalline cohomology class lifts if and only if $\bigoplus \mathrm{H}^i(X, W\Omega^i_{X,\log})[1/p] \to \bigoplus \mathrm{H}^i(Y, W\Omega^i_{Y,\log})[1/p]$ lifts. Thus we can apply (b) above to conclude the proof of the "local theorem".

The paper also contains useful applications and remarks. For example, it is shown that the Tate conjecture for divisors follows from its surface version; remarks on the existence of Leray spectral sequences for rigid cohomology in a certain circumstance are given. The article ends with an appendix explaining and clarifying the notion of "direct image F-isocrystal" in the sense of *A. Ogus* [Duke Math. J. 51, 765–850 (1984; Zbl 0584.14008)].

Reviewer: Dingxin Zhang (Beijing)

MSC:

14D15 Formal methods and deformations in algebraic geometry
19E15 Algebraic cycles and motivic cohomology (K-theoretic aspects)

Cited in **5** Documents

Keywords:

variational Hodge; Tate conjecture; topological cyclic homology; crystalline cohomology; rigid cohomology; de Rham-Witt complexes

Full Text: DOI arXiv

References:

[1] Ambrosi, E.: A note on the behaviour of the Tate conjecture under finitely generated field extensions.arXiv:1810.06480(2018)

[2] Ambrosi,E.:SpecializationofN´eron-Severigroupsinpositivecharacteristic. arXiv:1810.06481(2018)

[3] Andr´e, Y.: Une introduction aux motifs. Motifs purs, motifs mixtes, p´eriodes. Soc. Math. France, Paris (2004)Zbl 1060.14001 MR 2115000

[4] Artin, M.: Supersingular K3 surfaces. Ann. Sci. ´Ecole Norm. Sup. (4) 7, 543-567 (1974) Zbl 0322.14014 MR 0371899 · Zbl 0322.14014

[5] Berthelot, P.: Cohomologie cristalline des sch´emas de caract´eristique p > 0. Lecture Notes in Math. 407, Springer, Berlin (1974)Zbl 0298.14012 MR 0384804 · Zbl 0298.14012

[6] Berthelot, P.: G´eom´etrie rigide et cohomologie des vari´et´es alg´ebriques de caract´eristique p. In: Introductions aux cohomologies p-adiques (Luminy, 1984), M´em. Soc. Math. France (N.S.) 23, 3, 7-32 (1986)Zbl 0606.14017 MR 0865810

[7] Berthelot, P.: Cohomologie rigide et cohomologie rigide ‘a supports propres, premi‘ere partie. Pr´epublication IRMAR 96-03 (1996)

[8] Berthelot, P., Breen, L., Messing, W.: Th´eorie de Dieudonn´e cristalline. II. Lecture Notes in Math. 930, Springer, Berlin (1982)Zbl 0516.14015 MR 0667344 · Zbl 0516.14015

[9] Berthelot, P., Grothendieck, A., Illusie, L.: Th´eorie des intersections et th´eor‘eme de Riemann- Roch. S´eminaire de G´eom´etrie Alg´ebrique du Bois-Marie 1966-1967 (SGA 6). Lecture Notes in Math. 225, Springer, Berlin (1971)Zbl 0218.14001

[10] Berthelot, P., Illusie, L.: Classes de Chern en cohomologie cristalline. C. R. Acad. Sci. Paris S´er. A-B 270, A1695-A1697, A1750-A1752 (1970)Zbl 0198.26201 MR 0269660 · Zbl 0198.26201

[11] Berthelot, P., Messing, W.: Th´eorie de Dieudonn´e cristalline. III. Th´eor‘emes d’´equivalence et de pleine fid´elit´e. In: The Grothendieck Festschrift, Vol. I, Progr. Math. 86, Birkh¨auser Boston, Boston, MA, 173-247 (1990)Zbl 0753.14041 MR 1086886 · Zbl 0753.14041

[12] Berthelot, P., Ogus, A.: Notes on Crystalline Cohomology. Princeton Univ. Press, Princeton, NJ, and Univ. of Tokyo Press, Tokyo (1978)Zbl 0383.14010 MR 0491705 · Zbl 0383.14010

[13] Berthelot, P., Ogus, A.: F -isocrystals and de Rham cohomology. I. Invent. Math. 72, 159-199 (1983)Zbl 0516.14017 MR 0700767 · Zbl 0516.14017

[14] Bloch, S., Esnault, H., Kerz, M.: Deformation of algebraic cycle classes in characteristic zero. Algebr. Geom. 1, 290-310 (2014)Zbl 1322.14024 MR 3238152 · Zbl 1322.14024

[15] Bloch, S., Esnault, H., Kerz, M.: p-adic deformation of algebraic cycle classes. Invent. Math. 195, 673-722 (2014)Zbl 1301.19005 MR 3166216 · Zbl 1301.19005

[16] Chiarellotto, B., Le Stum, B.: Sur la puret´e de la cohomologie cristalline. C. R. Acad. Sci. Paris S´er. I Math. 326, 961-963 (1998)Zbl 0936.14016 MR 1649945 · Zbl 0936.14016

[17] Christensen, A.: Specialization of N´eron-Severi groups in characteristic p.arXiv:1810.06550 (2018)

[18] Davis, C., Langer, A., Zink, T.: Overconvergent de Rham-Witt cohomology. Ann. Sci. ´Ecole Norm. Sup. (4) 44, 197-262 (2011)Zbl 1236.14025 MR 2830387 · Zbl 1236.14025

[19] de Jong, A. J.: Tate conjecture for divisors. Unpublished note

[20] de Jong, A. J.: Smoothness, semi-stability and alterations. Inst. Hautes ´Etudes Sci. Publ. Math. 83, 51-93 (1996)Zbl 0916.14005 MR 1423020 · Zbl 0916.14005

[21] de Jong, A. J.: On a result of Artin.www.math.columbia.edu/⊠dejong/(2011)

[22] de Jong, A. J., Katz, N. M.: Monodromy and the Tate conjecture: Picard numbers and Mordell-Weil ranks in families. Israel J. Math. 120, part A, 47-79 (2000)Zbl 1067.14504 MR 1815370 · Zbl 1067.14504

[23] Deligne, P.: Th´eor‘eme de Lefschetz et crit‘eres de d´eg´en´erescence de suites spectrales. Publ. Math. Inst. Hautes ´Etudes Sci. 35, 107-126 (1968)Zbl 0159.22501 MR 0244265 · Zbl 0159.22501

[24] Deligne, P.: Th´eorie de Hodge. II. Publ. Math. Inst. Hautes ´Etudes Sci. 40, 5-57 (1971) Zbl 0219.14007 MR 0498551 · Zbl 0219.14007

[25] Deligne, P.: La conjecture de Weil. II. Publ. Math. Inst. Hautes ´Etudes Sci. 52, 137-252 (1980) Zbl 0456.14014 MR 0601520

[26] Dundas, B., Morrow, M.: Finite generation and continuity of topological Hochschild and cyclic homology. Ann. Sci. ´Ecole Norm. Sup. (4) 50, 201-238 (2017)Zbl 1372.19002 MR 3621430 · Zbl 1372.19002

[27] Geisser, T.: Motivic cohomology, K-theory and topological cyclic homology. In: Handbook of K-theory, Vol. 1, Springer, Berlin, 193-234 (2005)Zbl 1113.14017 MR 2181824 · Zbl 1113.14017

[28] Geisser, T., Hesselholt, L. Topological cyclic homology of schemes. In: Algebraic K-theory (Seattle, WA, 1997), Amer. Math. Soc., Providence, RI, 41-87 (1999)Zbl 0953.19001 MR 1743237 · Zbl 0953.19001

[29] Gillet, H., Messing, W.: Cycle classes and Riemann-Roch for crystalline cohomology. Duke Math. J. 55, 501-538 (1987)Zbl 0651.14014 MR 0904940 · Zbl 0651.14014

[30] Gros, M.: Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique. M´em. Soc. Math. France (N.S.) 21, 1-87 (1985)Zbl 0615.14011 MR 0844488 · Zbl 0615.14011

[31] Grothendieck, A.: ´El´ements de g´eom´etrie alg´ebrique. III. ´Etude cohomologique des faisceaux coh´erents. I. Publ. Math. Inst. Hautes ´Etudes Sci. 11, 5-167 (1961)Zbl 0118.36206 MR 0217085

[32] Grothendieck, A.: On the de Rham cohomology of algebraic varieties. Publ. Math. Inst. Hautes ´Etudes Sci. 29, 95-103 (1966)Zbl 0145.17602 MR 0199194 · Zbl 0145.17602

[33] Grothendieck, A.: Cohomologie locale des faisceaux coh´erents et th´eor‘emes de Lefschetz locaux et globaux (SGA 2). North-Holland, Amsterdam, and Masson, Paris (1968) Zbl 0197.47202 MR 2171939

[34] Grothendieck, A.: Groupes de monodromie en g´eom´etrie alg´ebrique. I. S´eminaire de G´eom´etrie Alg´ebrique du Bois-Marie 1967-1969 (SGA 7 I). Lecture Notes in Math. 288, Springer, Berlin (1972)Zbl 0237.00013 MR 0354656

[35] Illusie, L.: Report on crystalline cohomology. In: Algebraic Geometry (Arcata, CA, 1974), Proc. Sympos. Pure Math. 29, Amer. Math. Soc., Providence, RI, 459-478 (1975) Zbl 0326.14005 MR 0393034

[36] Illusie, L.: Complexe de de Rham-Witt et cohomologie cristalline. Ann. Sci. ´Ecole Norm. Sup. (4) 12, 501-661 (1979)Zbl 0436.14007 MR 0565469 · Zbl 0436.14007

[37] Jannsen,U.:Continuous´etalecohomology.Math.Ann.280,207-245(1988) Zbl 0649.14011 MR 0929536

[38] Katz, N. M.: Slope filtration of F -crystals. In: Journ´ees de G´eom´etrie Alg´ebrique de Rennes (Rennes, 1978), Vol. I, Ast´erisque 63, 113-163 (1979)Zbl 0426.14007 MR 0563463

[39] Katz, N. M., Messing, W.: Some consequences of the Riemann hypothesis for varieties over finite fields. Invent. Math. 23, 73-77 (1974)Zbl 0275.14011 MR 0332791 · Zbl 0275.14011

[40] Kedlaya, K. S.: Full faithfulness for overconvergent F -isocrystals. In: Geometric Aspects of Dwork Theory, Vol. II, de Gruyter, Berlin, 819-835 (2004)Zbl 1087.14018 MR 2099088 · Zbl 1087.14018

[41] Kedlaya, K. S.: Finiteness of rigid cohomology with coefficients. Duke Math. J. 134, 15-97 (2006)Zbl 1133.14019 MR 2239343 · Zbl 1133.14019

[42] Kedlaya, K. S.: Good formal structures for flat meromorphic connections, I: surfaces. Duke Math. J. 154, 343-418 (2010)Zbl 1204.14010 MR 2682186 · Zbl 1204.14010

[43] Kunz, E.: Characterizations of regular local rings for characteristic p. Amer. J. Math. 91, 772-784 (1969)Zbl 0188.33702 MR 0252389 · Zbl 0188.33702

[44] Lazda, C.: Incarnations of Berthelot's conjecture. J. Number Theory 166, 137-157 (2016) Zbl 1375.14078 MR 3486269 · Zbl 1375.14078

[45] Lazda, C., P´al, A.: Cycle classes in overconvergent rigid cohomology and a semistable Lefschetz (1, 1) theorem. Compos. Math. 155, 1025-1045 (2019)Zbl 07053890 MR 3946283 · Zbl 1440.14032

[46] Matsuda, S., Trihan, F.: Image directe sup´erieure et unipotence. J. Reine Angew. Math. 569, 47-54 (2004)Zbl 1105.14019 MR 2055713 · Zbl 1105.14019

[47] McCarthy, R.: Relative algebraic K-theory and topological cyclic homology. Acta Math. 179, 197-222 (1997)Zbl 0913.19001 MR 1607555 · Zbl 0913.19001

[48] Milne, J. S.: ´Etale Cohomology. Princeton Math. Ser. 33, Princeton Univ. Press, Princeton, NJ (1980)Zbl 0433.14012 MR 0559531

[49] Morrow, M. A case of the deformational Hodge conjecture via a pro Hochschild-Kostant- Rosenberg theorem. C. R. Math. Acad. Sci. Paris 352, 173-177 (2013)Zbl 1295.19002 MR 3167562 · Zbl 1295.19002

[50] Morrow, M.: K-theory and logarithmic Hodge-Witt sheaves of formal schemes in characteristic p. Ann. Sci. ´Ecole Norm. Sup. (4), to appear

[51] Ogus, A.: F -isocrystals and de Rham cohomology. II: Convergent isocrystals. Duke Math. J. 51, 765-850 (1984)Zbl 0584.14008 MR 0771383 · Zbl 0584.14008

[52] Ogus, A.: The convergent topos in characteristic p. In: The Grothendieck Festschrift, Vol. III, Progr. Math. 88, Birkh¨auser Boston, Boston, MA, 133-162 (1990)Zbl 0728.14020 MR 1106913 · Zbl 0728.14020

[53] Petrequin, D.: Classes de Chern et classes de cycles en cohomologie rigide. Bull. Soc. Math. France 131, 59-121 (2003)Zbl 1083.14505 MR 1975806 · Zbl 1083.14505

[54] Popescu, D.: General N´eron desingularization. Nagoya Math. J. 100, 97-126 (1985) Zbl 0561.14008 MR 0818160 · Zbl 0561.14008

[55] Popescu, D.: General N´eron desingularization and approximation. Nagoya Math. J. 104, 85- 115 (1986)Zbl 0592.14014 MR 1060701 · Zbl 0592.14014

[56] Shiho, A.: Crystalline fundamental groups. II: Log convergent cohomology and rigid cohomology. J. Math. Sci. Univ. Tokyo 9, 1-163 (2002)Zbl 1057.14025 MR 1889223 · Zbl 1057.14025

[57] Shiho, A.: On logarithmic Hodge-Witt cohomology of regular schemes. J. Math. Sci. Univ. Tokyo 14, 567-635 (2007)Zbl 1149.14013 MR 2396000 · Zbl 1149.14013

[58] Shiho, A.: Relative log convergent cohomology and relative rigid cohomology I. arXiv:0707.1742(2007)

[59] Tate, J.: Conjectures on algebraic cycles in ‘-adic cohomology. In: Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math. 55, Part 1, Amer. Math. Soc., Providence, RI, 71-83 (1994) Zbl 0814.14009 MR 1265523 · Zbl 0814.14009