Morrow, Matthew


The article under review proves a crystalline version of the variational Tate conjecture for divisors. Here is the precisely statement. Let $f: X \to S$ be a smooth projective morphism of smooth varieties over a perfect field of characteristic $p > 0$, let $s \in S$ be a closed point. Let $c \in H^2_{cris}(X/W(k))[1/p]$ be the cycle class of an element in $CH^1(X_s) \otimes \mathbb{Q}$. Then $c$ falls in the image of the restriction map $H^2_{cris}(X/W(k))[1/p] \to H^2_{cris}(X_s/W(k))[1/p]$ (c is “horizontal”) if and only if there exists $z \in CH^1(X) \otimes \mathbb{Q}$ such that $cl(z_{X_s}) = c$.

The theorem is a consequence of its local analogue (when $S$ is the spectrum of $A = k[t_1, \ldots, t_m]$), discussed in Section 3. We give a rough sketch of the proof of this local theorem. Set $Y_s = X \otimes A/(t)^{r+1}$, $Y = Y_0$.

(a) The trace map from the K-theory spectrum to the topological cyclic homology spectrum induces a homotopy pullback

$$\text{holim} K(Y_t) = TC(X;p) \times_{TC(Y;p),\pi} K(Y).$$

which induces a ladder of exact sequences of homotopy groups. This is due to [B. Dundas and M. Morrow, Ann. Sci. École Norm. Sup. (4) 50, 201-238 (2017; Zbl 1372.19002)]. The $\pi_0 \otimes \mathbb{Q}$ of the topological cyclic homology is the sum of $H^i(X, W \Omega_{X, log})[1/p]$ (the continuous cohomology of logarithmic de Rham-Witt sheaves, see [A. Shiho, J. Math. Sci., Tokyo 14, No. 4, 567–635 (2007; Zbl 1149.14013)])

(b) Spelling out the meaning of the exactness at $\pi_0$, one concludes that a class $z \in K_0(Y_t)$ lifts to $\lim_n K_0(Y_t)[1/p]$ if and only if $\text{ch}^{log}(z) \in \bigoplus H^i(Y, W \Omega_{Y, log})[1/p]$ lifts to a class in $\bigoplus H^i(X, W \Omega_{X, log})[1/p]$

(c) One shows, by studying the canonical map $\varepsilon: H^i(X, W \Omega_{X, log})[1/p] \to H^i_{cris}(X/W(k))[1/p]$, that the desired crystalline cohomology class lifts if and only if $\bigoplus H^i(X, W \Omega_{X, log})[1/p] \to \bigoplus H^i(Y, W \Omega_{Y, log})[1/p]$ lifts. Thus we can apply (b) above to conclude the proof of the “local theorem”.

The paper also contains useful applications and remarks. For example, it is shown that the Tate conjecture for divisors follows from its surface version; remarks on the existence of Leray spectral sequences for rigid cohomology in a certain circumstance are given. The article ends with an appendix explaining and clarifying the notion of “direct image $F$-isocrystal” in the sense of $A$. Ogus [Duke Math. J. 51, 765–850 (1984; Zbl 0584.14008)].

Reviewer: Dingxin Zhang (Beijing)

MSC:

14D15 Formal methods and deformations in algebraic geometry

19E15 Algebraic cycles and motivic cohomology ($K$-theoretic aspects)

Keywords:

variational Hodge; Tate conjecture; topological cyclic homology; crystalline cohomology; rigid cohomology; de Rham-Witt complexes

Full Text: DOI arXiv

References:


