Let $G$ be a complex reductive group with parabolic subgroup $P$. Let $n$ denote the nilpotent radical of the Lie algebra of $P$. For a nilpotent element $x$ in the Lie algebra of $G$, the associated Spaltenstein variety is

$$X^P_x := \{ gP \in G/P \mid g^{-1}xg \in n \}.$$

In general, a Spaltenstein variety is not irreducible, and a longstanding question is whether they are pure dimensional (or equidimensional). This is known to hold in the case that $P$ is a Borel subgroup or if $G$ is the general linear group. On the other hand, N. Spaltenstein [Classes unipotentes et sous-groupes de Borel. Berlin-Heidelberg-New York: Springer-Verlag (1982; Zbl 0486.20025)] constructed a counterexample for a special orthogonal group. The main result of this paper is that $X^P_x$ is pure dimensional if $G$ is classical and $x$ is an even or odd partition.

In fact, a stronger result is shown. Consider a Richardson element $e$ of the Lie algebra of $G$ and the associated nilpotent orbit $O_e$. There is a partial resolution of singularities $\pi: T^*(G/P) \to O_e$, where $T^*(G/P)$ denotes the cotangent bundle of $G/P$. Consider the Slodowy slice $S_e$ associated to $e$ and set $S_{e,x} = \pi^{-1}(S_{e,x})$. Then the author shows that $X^P_x$ is Lagrangian in $S_{e,x}$, giving the (pure) dimension of $X^P_x$ to be $\frac{1}{2} \dim T^*(G/P) - \frac{1}{2} \dim O_e$.

This is done by working in a symplectic geometry setting. A key component is a result of V. Ginzburg [Sémin. Congr. 24, 145–219 (2012; Zbl 1305.16009)] involving $\mathbb{C}^*$-actions on varieties. Given a smooth symplectic algebraic variety $\bar{Y}$ and affine variety $Y$ with $\mathbb{C}^*$-actions, along with a proper morphism $p: \bar{Y} \to Y$ that preserves the action, Ginzburg’s result gives conditions under which the inverse of the $\mathbb{C}^*$-fixed point locus of $Y$ is Lagrangian in $\bar{Y}$. The author frames the problem of interest in a more general context, allowing the use of Nakajima quiver varieties and their $\sigma$-quiver variants; a notion introduced in earlier work of the author [Represent. Theory 23, 1–56 (2019; Zbl 1403.16009)]. Lastly the author illustrates the result with several examples, including a renewed look at examples considered in the aforementioned work of Spaltenstein.

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