The minimal model program for deformations of Hilbert schemes of points on the projective plane. (English) Zbl 1432.14011

The minimal model program for the Hilbert schemes \( \text{Hilb}^n \mathbb{P}^2 \) of \( n \)-points on the complex projective plane \( \mathbb{P}^2 \) and the correspondence between wall-crossings in effective cones of divisors and wall-crossings in Bridgeland stability manifolds are explored in detail in [D. Arcara et al., Adv. Math. 235, 580–626 (2013; Zbl 1267.14023)]. The paper under review studies the relevant problems in ibid. in a broader setting.

The Sklyanin algebra \( S = \text{Skl}(E, \mathcal{L}, \lambda) \) is in general a noncommutative algebra, constructed from the data: an elliptic curve \( E \), a degree 3 line bundle \( \mathcal{L} \) on \( E \) and a translation \( \lambda \) on \( E \). When \( \lambda \) is chosen to be the identity, \( S \cong \mathbb{C}[x_0, x_1, x_2] \), the homogeneous coordinate ring of \( \mathbb{P}^2 \). The category of qgr-\( S \) (see Section 2) corresponds to, in commutative case, the category of coherent sheaves on \( \mathbb{P}^2 \). One can introduce Chern classes for the objects in qgr-\( S \) and hence the notion of slope stability. The moduli space \( \text{Hilb}^n S \) of semistable objects with variants \( (\text{rk}, c_1, \chi) = (1, 0, 1 - n) \) gives a generic deformation of \( \text{Hilb}^n \mathbb{P}^2 \) by [T. A. Nevins and J. T. Stafford, Adv. Math. 210, No. 2, 405–478 (2007; Zbl 1116.14003)] and [N. Hitchin, Mosc. Math. J. 12, No. 3, 567–591 (2012; Zbl 1267.32010)].

In a similar fashion to Bridgeland, Arcara-Bertram’s construction for stability conditions on projective surfaces, the authors construct stability conditions \( \sigma = \sigma(s, t) \) on the derived category \( D^b(\text{qgr-}S) \), for \( (s, t) \) in the upper-half plane. Let \( \mathcal{M}_\sigma(n) \) be the moduli space of stable objects of invariants \( (\text{rk}, c_1, \chi) = (1, 0, 1 - n) \) with respect to \( \sigma \). The authors show that \( \mathcal{M}_\sigma(n) \) coincides with \( \text{Hilb}^n S \) for \( s < 0 \) and \( t \gg 0 \).

The proof involves an interpretation for the moduli space as a moduli space of quiver representations using G.I.T.

One main theorem is that for general \( \sigma \) not lying on destabilizing walls, \( \mathcal{M}_\sigma(n) \) is a nonsingular, projective variety of dimension \( 2n \) so long as it is nonempty, and that for \( \sigma \neq \sigma' \) not on any destabilizing wall, \( \mathcal{M}_\sigma(n) \) and \( \mathcal{M}_{\sigma'}(n) \) are birationally equivalent.

Fix \( s \in \mathbb{R} \), then for the heart of the bounded t-structure corresponding to \( s \), one can let the parameter \( t \) decrease. When \( t \) crosses a destabilizing wall in the upper-half plane, there are induced birational transforms between the moduli spaces corresponding to \( \sigma(s, t) \) on the two sides of the wall. By the variation of G.I.T., the authors establish for \( \text{Hilb}^n S \) a one-to-one correspondence between the destabilizing walls in the upper-half plane of stability conditions and the stable base locus walls in the effective cone of divisors. In particular, for \( \text{Hilb}^n \mathbb{P}^2 \), an explicit correspondence between the destabilizing walls in the second quadrant and the stable base locus walls is confirmed, as conjectured by Arcara, Bertram, Coskun and Huizenga in [Zbl 1267.14023].