**Axler, Sheldon**
**Measure, integration & real analysis.** (English) Zbl 1435.28001
Graduate Texts in Mathematics 282. Cham: Springer (ISBN 978-3-030-33142-9/hbk; 978-3-030-33143-6/ebook). xviii, 411 p., open access (2020).

The book is a perfect introduction to graduate students into the theory of measure and Lebesgue integration together with some topics in Real Analysis. It is written in a very pedagogical way providing in each chapter many examples and a long collection of problems. The presentation is a gentle approach to serious mathematics with many examples and detailed proofs. Also, as a visual aid, throughout the book definitions are in yellow boxes and theorems are in blue boxes and each theorem has an informal descriptive name. There are also two prefaces, one for students and other for instructors which may be helpful. The book has, in my opinion, two different parts and it can be taught in two different semesters. The first part covering the first five chapters mostly devoted to measure and integration theory while the second one consisting on the content of chapters from six to twelve covering several topics in real analysis such as Banach and Hilbert spaces (and linear operators acting on them) and other important topics such as $L^p(\mu)$, Fourier transform, complex measures and probability.

The book is well organized and introduces the student into the topic in a progressive and motivated way. For instance, Chapter 1 starts with a brief review of Riemann integration and a discussion of the deficiencies of the Riemann integral which motivated the need of a better theory of integration. Chapter 2 begins defining outer measure on $\mathbb{R}$ as an extension of the length of an interval, and shows that it is not additive for general subsets in $\mathbb{R}$. This leads to restrict the attention to Borel sets and furthermore the general notion of $\sigma$-algebra or the $\sigma$-algebra generated for a family of sets. The definition of Lebesgue measurable set is given as the set which differs from a Borel set in a set of outer measure 0. Examples such as the Cantor set and the Cantor function are provided to separate the notion of Borel and Lebesgue measurability. With the notion of measurable functions some theorems of pointwise convergence, such as Egorov or Luzin theorems are given in this chapter. The notion of integration with respect to a measure appears in Chapter 3. It is done starting first for nonnegative measurable functions. The classical results such as the Monotone Convergence Theorem and the Dominated Convergence Theorem which allow to interchange limits and integrals under appropriate conditions are presented. Chapter 4 is devoted to the use of the Lebesgue Differentiaton Theorem, whose proof is done by means of the Hardy-Littlewood maximal inequality. Also the application to the Lebesgue Density Theorem is provided. Product measures is the main topic of Chapter 5. Besides the construction of the product of measure spaces and Tonelli and Fubini Theorems the chapter contains results on integration on $\mathbb{R}^n$ as a product measure and applications to compute the volume of the unit ball of $\mathbb{R}^n$ and the equality of mixed partial derivatives via Fubini's Theorem.

The book now changes a bit the perspective. Chapter 6 contains a quick review of metric and normed spaces. Introducing to Banach space theory and its key results such as Hahn-Banach Theorem and the Baire's Theorem and its consequences (Open Mapping Theorem, Inverse Mapping Theorem, Closed Graph Theorem and Principle of Uniform Boundedness). Chapter 7 is devoted to the study of $L^p(\mu)$ spaces for $1 \leq p \leq \infty$ having the Hölder's inequality and Minkowski's inequality as major tools and providing a proof of the duality $(\ell_p)^* = \ell_{p'}$ for $1 \leq p < \infty$ and $1/p + 1/p' = 1$. The rest of the monograph, as mentioned by the author, needs not be covered in order and chapters can be interchanged since they are of independent nature.

Chapter 8 deals with Hilbert spaces and contains the basic theory, namely Cauchy-Schwarz inequality, orthogonal projections, Riesz representation theorem and the notion of orthonormal basis with the corresponding Bessel inequality, Parseval's identity and Gram-Schmidt process. So far only positive measures had appeared in the monograph, the author uses Chapter 9 to introduce real and complex measures and the notion of total variation. The chapter contains the basic results in the theory covering the Hahn Decomposition Theorem, the Jordan Decomposition Theorem, the Lebesgue Decomposition Theorem and the Radon-Nikodym theorem whose proof is presented following von Neumann's idea using Hilbert spaces. Once this result is proved the duality $(L^p(\mu))^* = L^{p'}(\mu)$ is easily achieved. Chapter 10 is devoted to operators on Hilbert spaces and covers the topics of Adjoints and Invertibility and the notion of

Spectrum. The author includes Compact Operators and proves the Fredholm Alternative and two major results: the Spectral Theorem for compact operators and the Singular Value Decomposition for compact operators. Chapter 11 gives a gentle but modern introduction to Fourier series and Fourier transform. It starts off with Fourier coefficients and summability kernels such as Poisson Kernel, used for the solution to the Dirichlet problem on the disk and uses convolution as a major tool. When working on the circle the author makes use of orthonormal basis considered in previous chapters and, in many occasions, he does not look for the full generality of the results, for instance Pointwise Convergence of Fourier series is proved only for twice continuously differentiable functions. The Fourier transform is considered in $\mathbb{R}$ and the objective is to show the Fourier Inversion Formula and to use it to get a unitary operator on $L^2(\mathbb{R})$.

The last chapter takes the advantage of the book's earlier development of measure theory to present the basic language of probability theory, emphasizing the concept of independence that distinguish probability theory from measure theory. The chapter discusses the concepts of standard deviation, conditional probability and distribution function. It includes a proof of the basic results such as Bayes' Theorem and Weak Law of Large Numbers.

The book will become an invaluable reference for graduate students and instructors. Those interested in measure theory and real analysis will find the monograph very useful since the book emphasizes getting the students to work with the main ideas rather than on proving all possible results and it contains a rather interesting selection of topics which makes the book a nice presentation for students and instructors as well.

Reviewer: Oscar Blasco (Valencia)

**MSC:**

28-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to measure and integration
26-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to real functions
42-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to harmonic analysis on Euclidean spaces
46-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to functional analysis
60-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to probability theory



**Keywords:**

Riemann integration; measures; Lebesgue measure; integration; Lebesgue differentiation theorem; product measures; Banach spaces; Hilbert spaces; Fourier analysis; probability; compact operator

**Full Text:** DOI Link