Gaitsgory, Dennis; **Lurie, Jacob**
Weil's conjecture for function fields: volume I. (English) Zbl 1439.14006
Annals of Mathematics Studies 199. Princeton, NJ: Princeton University Press (ISBN 978-0-691-18213-1/hbk; 978-0-691-18214-8/pbk; 978-0-691-18443-2/ebook). viii, 311 p. (2019).

This monograph, together with its sequel, establishes Weil's Conjecture on Tamagawa numbers in the function field case. The proof is very conceptional and relies on the moduli stack of G-bundles and higher category theory. Certain steps in the proof are deferred to the second volume. The book is written in a clear and vivid style, pays attention to foundations and details, and yet elucidates motivations and ideas. It should be highly useful for researchers working with stacks and higher category theory.

Let K be a global field and G be an algebraic group that is connected and semisimple. The ring of adeles $K \subset \mathbb{A}_K \subset \prod_\nu K_\nu$ is locally compact and comes with a normalized Haar measure. Likewise, the group of adelic points $G(\mathbb{A})$ comes with a canonical choice of Haar measure μ_{Tam} called the *Tamagawa measure*. The volume of the quotient by the discrete subgroup $G(K)$ is the *Tamagawa number* $\tau(G)$.

Weil computed $\tau(\text{SO}_q) = 2$ and observed in many examples that $\tau(G) = 1$ if the algebraic group is simply-connected [*A. Weil*, Adèles and algebraic groups. (Appendix 1: The case of the group G_2, by M. Demazure. Appendix 2: A short survey of subsequent research on Tamagawa numbers, by T. Ono). Basel: Birkhäuser/Springer (1982; Zbl 0493.14028)]. The latter became known as Weil's Conjecture on Tamagawa numbers. It was proved by Kottwitz for number fields [*R. E. Kottwitz*, Ann. Math. (2) 127, No. 3, 629–646 (1988; Zbl 0678.22012)] The missing case E_8 also holds true, by work of *V. I. Chernousov* [Sov. Math., Dokl. 39, No. 3, 592–596 (1989; Zbl 0703.20040); translation from Dokl. Akad. Nauk SSSR 306, No. 5, 1059–1063 (1989)].

This monograph tackles the case of function fields, where K corresponds to an algebraic curve X of genus $g \geq 0$ over some finite field $\mathbb{F}_q$. The places ν can be identified with closed points $x \in X$. The ring of adeles, now written as $\mathbb{A}_X$, comes with the compact open subring of integral adeles $\mathbb{A}_X^\emptyset$. In this setting, notation is changed: $G \to X$ denotes an integral model of the algebraic group G_0 over K. Up to normalization, the volume of $G(\mathbb{A}_X^\emptyset)$ becomes an infinite product over the closed points $x \in X$, with factors $|G(\kappa(x)|/|\kappa(x)|^{n+\nu_x(\omega)}$, where ω is a non-zero left-invariant differential and $n = \dim(G_0)$. Using that the double coset $G(K_X)\backslash G(\mathbb{A}_X/G(\mathbb{A}_X^\emptyset)$ parameterizes isomorphism classes of G-torsors, one may rewrite Weil's Conjecture as mass formula:

$$|\operatorname{Tors}_G(X)| = q^{n(g-1)+\deg(\omega_{G/X})} \prod_{x\in X} \frac{|G(\kappa(x)|}{|\kappa(x)|^{n+\nu_x(\omega)}}.$$

Here the left-hand side is the mass $\sum \frac{1}{|\operatorname{Aut}(\mathcal{P}|}$ for the groupoid of G-torsors $\mathcal{P}$, which is a fiber category in the corresponding Artin stack Bun_G. Note that the convergence of the infinite sum is part of the assertion.

With a suitable form of the Grothendieck-Lefschetz Trace Formula for Bun_G, the mass formula can be rewritten in cohomological form:

$$\operatorname{Tr}(\operatorname{Frob}^{-1} | H^*(\operatorname{Bun}_G(\overline{X}, \mathbb{Z}_\ell)) = \prod_{x\in X} \frac{|G(\kappa(x)|}{|\kappa(x)|^{n+\nu_x(\omega)}}.$$

The book and its sequel are devoted to the proof of this. The main ingredient is the construction, and identification in the derived category, of ℓ-adic cochain complexes

$$\bigotimes_{x\in\overline{X}} C^*(BG_x, \mathbb{X}_\ell) = C^*(\operatorname{Bun}_G(\overline{X}, \mathbb{Z}_\ell)),$$

where the left-hand side is a continuous tensor product.

The book has five chapters. The first is a comprehensive introduction, containing the history of Weil's Conjecture, a discussion of various examples and reformulations, and an outline of the proof. Chapter 2 contains a review of ℓ-adic sheaves, which emphasizes the role triangulated categories and higher category theory. In Chapter 3, the authors construct for each Artin stack $\mathcal{Y}$ over a ground field k an ℓ-adic cochain complex $C^*(\mathcal{Y}, \mathbb{Z}_\ell)$, which is viewed as an $\mathbb{E}_\infty$-algebra. The latter structure is crucial to make sense of the tensor product $\bigotimes_{x\in\overline{X}} C^*(BG_x, \mathbb{Z}_\ell)$. Chapter 4 is devoted to the proof of Weil's Conjecture in cohomological form, by constructing a canonical map

$$\int_{\overline{X}} [BG]_{\overline{X}} \longrightarrow C^*(\mathrm{Bun}_G(\overline{X}, \mathbb{Z}_\ell).$$

The last chapter contains a proof of the Grothendieck–Lefschetz Trace Formula

$$q^{-\dim(\mathrm{Bun}_G(X)}|\operatorname{Tors}_G(X)| = \operatorname{Tr}(\mathrm{Frob}^{-1}|H^*(\mathrm{Bun}_G(\overline{X}, \mathbb{Z}_\ell)),$$

which gives the link between the cohomological form of Weil's Conjecture and the mass formula.

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MSC:

Cited in **9** Documents

14-02	Research exposition (monographs, survey articles) pertaining to algebraic geometry
14F30	p-adic cohomology, crystalline cohomology
11R58	Arithmetic theory of algebraic function fields
14H60	Vector bundles on curves and their moduli
14G10	Zeta functions and related questions in algebraic geometry (e.g., Birch-Swinnerton-Dyer conjecture)
14H05	Algebraic functions and function fields in algebraic geometry
14D20	Algebraic moduli problems, moduli of vector bundles

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