In this paper, the author studies $K$-theoretic Donaldson invariants which are holomorphic Euler characteristics of determinant line bundles on moduli spaces of rank-2 sheaves on surfaces. Let $(X, \omega)$ be a pair consisting of a rational surface $X$ and an ample line bundle $\omega$ on $X$. Let $M^X_{c_1}(d)$ be the moduli space of $\omega$-semistable torsion-free coherent sheaves of rank-2 on $X$ with Chern classes $c_1 \in H^2(X, \mathbb{Z})$ and $c_2 \in \mathbb{Z}$ such that $d = 4c_2 - c_1^2$. Associated to a line bundle $L$ on $X$, there is a determinant line bundle $\mu(L)$ on $M^X_{c_1}(d)$. Let $\Lambda$ be a formal variable. The goal of the paper is to study the generating function

$$
\chi^X_{c_1, \omega}(L) = \sum_{d \geq 0} \chi(M^X_{c_1}(d), \mu(L)) \Lambda^d
$$

of the holomorphic Euler characteristics $\chi(M^X_{c_1}(d), \mu(L))$. Assume that $\omega \cdot K_X < 0$ where $K_X$ is the canonical divisor of $X$, and that $\omega = H - a_1 E_1 - \ldots - a_n E_n$ with each $a_i < 1/\sqrt{n}$ when $X$ is the blown-up of the projective plane $\mathbb{P}^2$ at $n$ points with exceptional divisors $E_1, \ldots, E_n$ and $H$ is a line in $\mathbb{P}^2$. The main theorem of the paper states that if $X$ is a rational surface, then there exist a polynomial $P^X_{c_1,L}(\Lambda) \in \Lambda^{-d_0} \mathbb{Q}[\Lambda^{d_0}]$ and a non-negative integer $l^X_{c_1,L}$ such that

$$
\chi^X_{c_1, \omega}(L) \equiv \frac{P^X_{c_1,L}(\Lambda)}{(1 - \Lambda^{d_0})^{l^X_{c_1,L}}}
$$

where for two Laurent series $P(\Lambda) = \sum_n a_n \Lambda^n, Q(\Lambda) = \sum_n b_n \Lambda^n \in \mathbb{Q}[\Lambda^{-1}][[\Lambda]],$ define $P(\Lambda) \equiv Q(\Lambda)$ if there exists an integer $n_0$ such that $a_n = b_n$ for all $n \geq n_0$. Based on some explicit calculations on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, the author proposed several interesting conjectures regarding $P^X_{c_1,L}(\Lambda)$ and $l^X_{c_1,L}$. These results are analogue to the Verlinde formula for algebraic curves, and related to Le Potier’s strange duality conjecture. The main ideas of the proof are to use the wall-crossing formula and blown-up formula for $\chi^X_{c_1, \omega}(L)$, and to analyze the $K$-theoretic Donaldson invariants with point class.

Section 2 is devoted to background materials such as determinant line bundles, walls and chambers, and $K$-theoretic Donaldson invariants. Section 3 reviews the strange duality conjecture for surfaces, and interprets the main results and conjectures in view of strange duality. Section 4 recalls Theta functions, modular forms and the wall-crossing formula. For the $K$-theoretic Donaldson invariants with point class, the polynomiality and vanishing of the wall-crossing formula are investigated. In Section 5, the author studies the $K$-theoretic Donaldson invariants for polarizations on the boundary of the ample cone. Section 6 applies blowup polynomials, blowup formulas and blowdown formulas to the present paper. Recursion formulas for rational ruled surfaces are proved in Section 7. Computations of the invariants for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are carried out in Section 8 and Section 9 respectively.

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MSC:

- 14J60 Vector bundles on surfaces and higher-dimensional varieties, and their moduli
- 14D21 Applications of vector bundles and moduli spaces in mathematical physics (twistor theory, instantons, quantum field theory)
- 14D22 Fine and coarse moduli spaces
- 14F08 Derived categories of sheaves, dg categories, and related constructions in algebraic geometry

Keywords:

- moduli of sheaves; determinant bundle; strange duality; Verlinde formula; Donaldson invariants
References:


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[16] Zagier, D.

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