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Verlinde-type formulas for rational surfaces. (English) Zbl 1442.14135

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In this paper, the author studies K-theoretic Donaldson invariants which are holomorphic Euler characteristics of determinant line bundles on moduli spaces of rank-2 sheaves on surfaces. Let (X, ω) be a pair consisting of a rational surface X and an ample line bundle ω on X. Let $M_\omega^X(c_1, d)$ be the moduli space of ω-semistable torsion-free coherent sheaves of rank-2 on X with Chern classes $c_1 \in H^2(X, \mathbb{Z})$ and $c_2 \in \mathbb{Z}$ such that $d = 4c_2 - c_1^2$. Associated to a line bundle L on X, there is a determinant line bundle $\mu(L)$ on $M_\omega^X(c_1, d)$. Let Λ be a formal variable. The goal of the paper is to study the generating function

$$\chi_{c_1}^{X,\omega}(L) = \sum_{d>0} \chi\big(M_\omega^X(c_1, d), \mu(L)\big)\Lambda^d$$

of the holomorphic Euler characteristics $\chi\big(M_\omega^X(c_1, d), \mu(L)\big)$. Assume that $\omega \cdot K_X < 0$ where K_X is the canonical divisor of X, and that $\omega = H - a_1E_1 - \ldots - a_nE_n$ with each $a_i < 1/\sqrt{n}$ when X is the blown-up of the projective plane $\mathbb{P}^2$ at n points with exceptional divisors $E_1, \ldots, E_n$ and H is a line in $\mathbb{P}^2$. The main theorem of the paper states that if X is a rational surface, then there exist a polynomial $P_{c_1,L}^X(\Lambda) \in \Lambda^{-c_1^2}\mathbb{Q}[\Lambda^{\pm 4}]$ and a non-negative integer $l_{c_1,L}^X$ such that

$$\chi_{c_1}^{X,\omega}(L) \equiv \frac{P_{c_1,L}^X(\Lambda)}{(1-\Lambda^4)^{l_{c_1,L}^X}}$$

where for two Laurent series $P(\Lambda) = \sum_n a_n\Lambda^n, Q(\Lambda) = \sum_n b_n\Lambda^n \in \mathbb{Q}[\Lambda^{-1}][[\Lambda]]$, define $P(\Lambda) \equiv Q(\Lambda)$ if there exists an integer n_0 such that $a_n = b_n$ for all $n \geq n_0$. Based on some explicit calculations on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, the author proposed several interesting conjectures regarding $P_{c_1,L}^X(\Lambda)$ and $l_{c_1,L}^X$. These results are analogue to the Verlinde formula for algebraic curves, and related to Le Potier's strange duality conjecture. The main ideas of the proof are to use the wall-crossing formula and blown-up formula for $\chi_{c_1}^{X,\omega}(L)$, and to analyze the K-theoretic Donaldson invariants with point class.

Section 2 is devoted to background materials such as determinant line bundles, walls and chambers, and K-theoretic Donaldson invariants. Section 3 reviews the strange duality conjecture for surfaces, and interprets the main results and conjectures in view of strange duality. Section 4 recalls Theta functions, modular forms and the wall-crossing formula. For the K-theoretic Donaldson invariants with point class, the polynomiality and vanishing of the wall-crossing formula are investigated. In Section 5, the author studies the K-theoretic Donaldson invariants for polarizations on the boundary of the ample cone. Section 6 applies blowup polynomials, blowup formulas and blowdown formulas to the present paper. Recursion formulas for rational ruled surfaces are proved in Section 7. Computations of the invariants for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are carried out in Section 8 and Section 9 respectively.

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MSC:

14J60 Vector bundles on surfaces and higher-dimensional varieties, and their moduli
14D21 Applications of vector bundles and moduli spaces in mathematical physics (twistor theory, instantons, quantum field theory)
14D22 Fine and coarse moduli spaces
14F08 Derived categories of sheaves, dg categories, and related constructions in algebraic geometry

Keywords:

moduli of sheaves; determinant bundle; strange duality; Verlinde formula; Donaldson invariants

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