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Lévy matters VI. Lévy-type processes: moments, construction and heat kernel estimates. (English) Zbl 1442.60002

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From the author's preface: "I obtained the results presented in this book during my PhD."

Although the set of all Lévy processes (i. e., the set of all stochastically continuous processes with stationary and independent increments or equivalently the set of all infinitely divisible distribution functions) including Brownian motion and the Poisson process turned out to be a very rich and fruitful one, particularly in relation to a stochastic modelling of observable phenomena in concrete applications which cannot be described by continuous paths of Brownian motion only, including applications in financial mathematics and physics (cf. [*W. A. Woyczyński*, in: Lévy processes. Theory and applications. Boston: Birkhäuser. 241–266 (2001; Zbl 0982.60043)], a drawback of Lévy processes is that they do not allow a modelling of the dynamics of stochastic processes which are not homogeneous in time and space.

A striking key result in theory and applications of Lévy processes (with trajectories in $\mathbb{R}^d$) is the famous Lévy-Itô decomposition. Let us recall that in the one-dimensional case ($d=1$) every (càdàg) Lévy process $L=(L_t)_{t\geq 0}$ can be written as

$$L_t = L_0 + \gamma\, t + \sqrt{q}\, B_t + C_t + \sum_{0\leq s<t} \Delta L_s\, \mathbf{1}_{\mathbb{R}\setminus[-1,1]}\left(\Delta L_s\right) ,$$

where $\gamma\in\mathbb{R}$, $q\geq 0$, $B=(B_t)_{t\geq 0}$ is a standard Brownian motion and $C=(C_t)_{t\geq 0}$ is a purely discontinuous martingale of compensated jumps of absolute size below 1 which is independent of B. $\Delta L_s := L_s - L_{s-}$ denotes the jump size at time s, so that the last sum represents the at most countably many jumps of the (càdàg version of the) process L with absolute jump size strictly larger than 1. A deeper analysis of the martingale part C reveals that it can be written as the stochastic integral

$$C_t = \int_0^t \int_{[-1,1]} x \left(J^L\left(ds,dx\right) - \nu^L(dx)\, ds\right) ,$$

where for all $t\geq 0$ and Borel sets $B\in\mathcal{B}(\mathbb{R})$ $J^L(\omega;[0,t]\times B)$ denotes a random jump measure, counting how many jumps of L of jump size within B occur for a fixed scenario ω in the time interval $[0,t]$, and $\nu^L(B) := \mathbb{E}\big[J^L(\cdot;[0,1]\times B)\big]$ is the average number of jumps of L of jump size within B in the time interval $[0,1]$. Similarly, in the general d-dimensional case the canonical representation for semimartingales implies that for every Lévy process $L=(L_t)_{t\geq 0}$ with paths in $\mathbb{R}^d$ there exists a – unique – triple (γ,Q,ν), the so called Lévy triple, consisting of a vector $\gamma\in\mathbb{R}^d$, a positive semidefinite matrix $Q=\sqrt{Q}^2\in\mathbb{R}^{d\times d}$ and the Lévy measure $\nu\equiv\nu^L$ such that

$$L_t = L_0 + \gamma\, t + \sqrt{Q}\, B_t + \int_0^t \int_{\|x\|\leq 1} x \left(J^L\left(ds,dx\right) - \nu(dx)\, ds\right) + \int_0^t \int_{\|x\|>1} x\, J^L\left(ds,dx\right) .$$

Another very important fact about Lévy processes $L=(L_t)_{t\geq 0}$ in $\mathbb{R}^d$ is that they are completely determined by the characteristic function

$$\varphi_{L_t}(x) := \widehat{\mathbb{P}_{L_t}} = \mathbb{E}\big[\exp\left(i\, x\, L_t\right)\big] \quad (x\in\mathbb{R}^d, t\geq 0) .$$

This follows from another famous result, namely the Lévy-Khinchin formula, stating in particular that if L is a Lévy process in $\mathbb{R}^d$ with Lévy triple (γ,Q,ν) then for all $x\in\mathbb{R}^d$ and $t\geq 0$

$$\varphi_{L_t}(x) = \exp\left(-t\, \psi_{(\gamma,Q,\nu)}(x)\right) ,$$

where

$$\psi_{(\gamma,Q,\nu)}(x) := -i\,\langle \gamma, x\rangle + \frac{1}{2}\langle x, Qx\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - \exp\left(i\,\langle \xi, x\rangle\right) + i\,\langle \xi, x\rangle\,\mathbf{1}_{[-1,1]}\left(\|\xi\|\right)\right)\,\nu(d\xi)\,.$$

$\psi_{(\gamma,Q,\nu)}$ is known as the characteristic exponent of the Lévy process L.

Much more is true. Namely, given any continuous negative definite function $f : \mathbb{R}^d \longrightarrow \mathbb{C}$, satisfying $f(0) = 0$ (cf. [*G. Forst*, Z. Wahrscheinlichkeitstheor. Verw. Geb. 34, 313–318 (1976; Zbl 0312.60009)] and [*N. Jacob*, Pseudo differential operators and Markov processes. In 3 vol. Vol. 1: Fourier analysis and semigroups. London: Imperial College Press (2001; Zbl 0987.60003)]) then there exists a Lévy process $Y = (Y_t)_{t\geq 0}$ with Lévy triple (β, Σ, μ) such that $f = \psi_{(\beta,\Sigma,\mu)}$ and hence $\varphi_{Y_t}(z) = \mathbb{E}\big[\exp\left(i\,z\,Y_t\right)\big] = \exp\left(-t\,f(z)\right)$ for all $z \in \mathbb{R}^d$ and $t \geq 0$.

In other words, the set of all Lévy processes is fully encoded in the set of all continuous negative definite functions $\psi : \mathbb{R}^d \longrightarrow \mathbb{C}$, satisfying $\psi(0) = 0$! The latter set has many non-trivial applications, also in mathematics itself including Harmonic Analysis on (abelian) semigroups (cf. [*C. Berg*, "Stieltjes-Pick-Bernstein-Schoenberg and their connection to complete monotonicity", in: Positive definite functions. From Schoenberg to space-time challenges. Ed. J. Mateu and E. Porcu. Dept. of Mathematics, University Jaume I, Castellon, Spain (2008)]). Here, any curious reader of this book should recall Bochner's Theorem, implying that in fact $\mathcal{M}^1_+\left(\mathbb{R}^d\right)$, the set of all probability measures on $\mathbb{R}^d$ maps one-to-one onto the set of all continuous positive definite functions $\kappa : \mathbb{R}^d \longrightarrow \mathbb{C}$, satisfying $\kappa(0) = 1$ – via the Fourier transform $\mathcal{M}^1_+\left(\mathbb{R}^d\right) \ni \mathbb{P} \mapsto \widehat{\mathbb{P}}$ and keep in their mind that a function $\psi : \mathbb{R}^d \longrightarrow \mathbb{C}$ is negative definite if and only if the function $\exp\left(-t\,\psi\right)$ is positive definite for all $t > 0$ (cf. [*C. Berg*, loc. cit.] and [*N. Jacob*, Pseudo differential operators and Markov processes. In 3 vol. Vol. 1: Fourier analysis and semigroups. London: Imperial College Press (2001; Zbl 0987.60003)]).

The author's transition from Lévy processes to Feller processes is realised by means of Markov semigroups, satisfying additional properties including the so called Feller property and strong continuity. This approach follows from the well-known fact that any Lévy process $L = (L_t)_{t\geq 0}$ in particular is a time-homogeneous Markov process with translation invariant transition functions

$$p_t\left(x, B\right) := \mathbb{P}^x\left(L_t \in B\right) := \mathbb{P}\left(L_t + x \in B\right), \quad (t \geq 0,\, x \in \mathbb{R}^d,\, B \in \mathcal{B}(\mathbb{R}^d))\,.$$

Let $\mathcal{C}_b(\mathbb{R}^d) \equiv \mathcal{C}_b(\mathbb{R}^d, \mathbb{R})$ (respectively $\mathcal{B}_b(\mathbb{R}^d) \equiv \mathcal{B}_b(\mathbb{R}^d, \mathbb{R})$) denote the space of all continuous, real-valued and bounded functions (respectively Borel measurable, real-valued and bounded functions) and $\mathcal{C}_\infty(\mathbb{R}^d) \equiv \mathcal{C}_\infty(\mathbb{R}^d, \mathbb{R})$ be the space of continuous and real-valued functions vanishing at infinity. Both spaces $\mathcal{C}_b(\mathbb{R}^d)$ and $\mathcal{C}_\infty(\mathbb{R}^d)$ are equipped with the norm $\|\cdot\|_\infty$ of uniform convergence respectively. If

$$P_t f(x) := \mathbb{E}^x f(L_t), \quad (t \geq 0, x \in \mathbb{R}^d,\, f \in \mathcal{B}_b(\mathbb{R}^d))$$

then the family $(P_t)_{t\geq 0}$ – induced by the Lévy process L – consists of a Markov semigroup of bounded linear operators $P_t : \mathcal{B}_b(\mathbb{R}^d) \longrightarrow \mathcal{B}_b(\mathbb{R}^d)$ which in addition satisfies $P_t(\mathcal{C}_\infty(\mathbb{R}^d)) \subseteq \mathcal{C}_\infty(\mathbb{R}^d)$ for all $t \geq 0$ (Feller property) and strong continuity on $\mathcal{C}_\infty(\mathbb{R}^d)$, i. e., $\lim\limits_{t\to\infty} \|P_t f - f\|_\infty = 0$ for all $f \in \mathcal{C}_\infty(\mathbb{R}^d)$.

By definition a Feller process is a Markov process whose Markov semigroup $(P_t)_{t\geq 0}$ in addition is strongly continuous on $\mathcal{C}_\infty(\mathbb{R}^d)$ and satisfies the Feller property, implying that every Lévy process is a Feller process. Feller processes satisfy very useful properties including the existence of cádlág modifications, the strong Markov property and right-continuity of the filtration.

The existence and construction of Feller processes is a major problem. There are many approaches: using the Hille-Yosida theorem and Kolmogorov's construction, solving the associated evolution equation (Kolmogorov's backwards equation), proving the well-posedness of the martingale problem, solving a stochastic differential equation. The conditions for these constructions are usually quite technical (cf. [*B. Böttcher*, "Feller processes: The next generation in modeling. Brownian motion, Lévy processes and beyond", PLoS ONE 5(12): e15102.2001, `doi:10.1371/journal.pone.0015102`]).

Particularly regarding existence and construction of Feller processes the author provides many important results, all centering around Theorem 3.3 – in my view, the author's main result at all. In Theorem 3.3 sufficient conditions on a subset of negative definite functions are listed, leading to an extension of the Lévy-Khinchin formula to Feller processes, respectively to their "symbol", where the latter generalises the characteristic exponent of a Lévy process (see Theorem 1.21 and Definition 1.22). However, as opposed to the Lévy process case, the set of all Feller processes is not fully encoded in the set of all symbols. As shown by the author in Example 3.1 there exists a family of space dependent continuous negative definite functions $q \equiv \{q(x, \cdot) : x \in \mathbb{R}^d\}$ such that its allocated "locally fixed" Lévy process family (Lévy-Khinchin, applied to each fixed $x \in \mathbb{R}^d$!) cannot arise from a Feller process with symbol q.

Following the author's own remark, Feller processes behave locally like Lévy processes (that is the reason why they are also called Lévy-type processes). However, their Lévy triplet depends on the current position of the Feller process.

Summing up the content, in Chapter 1 of the book an introduction to the objects the author is dealing with is given in detail including the introduction of Feller processes and their symbol, the martingale problem and the important parametrix construction on which the proof of Theorem 3.3 (in Chapter 4.8) is built. In Chapter 2 distributional properties of Feller processes are analysed thoroughly, concentrating on the existence of generalised moments and moment estimates. In chapters 3 and 4 – the core of the book – the non-trivial existence and construction of Feller processes is thoroughly illuminated and discussed in depth, yet without neglecting a discussion of some drawbacks and their implications (and hence open problems for researchers in this field), such as the seemingly crucial assumption of rotational invariance in dimension $d > 1$ in Theorem 3.3. In the final chapter, Chapter 5, concrete examples of Feller processes as applications of Theorem 3.3. are given. In particular, again by making use of the parametrix construction, a new existence and uniqueness result for stochastic differential equations, driven by Lévy processes is provided.

It is for sure that I can highly recommend to read this book which contains many interesting results on a fascinating, technically demanding topic, carefully analysed and discussed by the author who links parts of probability theory, stochastic processes, Fourier analysis and positivity (in the sense of Schoenberg) deeply. The book should also be studied by pure analysts who are keen to check whether some of the given results in the book can be derived by purely functional analytic methods – shedding further light on an understanding of Feller processes.

Reviewer: Frank Oertel (Rosenheim)

MSC:

60-02 Research exposition (monographs, survey articles) pertaining to probability theory

60G51 Processes with independent increments; Lévy processes

35K08 Heat kernel

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