Algebraic varieties equipped with a reductive group action are considered. The goal is to generalize the Białynicki-Birula decomposition for torus action [A. Białynicki-Birula, Ann. Math. (2) 98, 480–497 (1973; Zbl 0275.14007)] to general situations, when a linearly reductive group acts. The paper deals with fundamental questions, it is clearly, very well written.

The original Białynicki-Birula theorem for $G = \mathbb{G}_m$ (the one dimensional torus) states that any smooth and complete algebraic $G$-variety $X$ can be decomposed into cells $X^+_i$ indexed by the components of $X^G$. Let $X^+$ be the disjoint union of the cells. The resulting natural maps

$$i_X : X^+ \to X \quad \text{the sum of inclusions}$$
$$\pi_X : X^+ \to X^G \quad \text{the limit map } \lim_{t \to 0} tx$$

are $G$-equivariant and $\pi_X$ is a locally trivial affine fibration. Moreover the action of $G$ on $X^+$ extends to an action of $\overline{G} = \mathbb{A}^1$. When we drop the assumption on $X$ being smooth and complete, then the map $i_X$ does not have to be a bijection on the closed points and $\pi_X$ is not necessarily a fibration. In the general setting for $G = \mathbb{G}_m$ the functoriality of $X^+$ was observed by V. Drinfeld [“On algebraic spaces with an action of $G_m$”, Preprint, arXiv:1308.2604].

The paper under review generalizes the situation to the following interesting and natural setting. Suppose $G$ is a connected linearly reductive affine group. Let $\overline{G}$ be a monoid containing $G$ as a dense set. The basic example is an affine toric variety or the monoid of $n \times n$ matrices containing $GL_n$. With a mild assumption on $X$ (locally of finite type) the scheme $X^+$ together with the map $i_X$ is defined. The scheme $X^+$ represents the functor $D_{X,\overline{G}} : \text{Sch}_k^{op} \to \text{Set}$

$$D_{X,\overline{G}}(S) = \{ \varphi : \overline{G} \times X \to X \mid \varphi \text{ is } G\text{-equivariant} \}.$$

If the monoid $\overline{G}$ contains a zero and $X$ is smooth then the limit map $\pi_X : X^+ \to X^G$ exists and it is an affine fibre bundle. The result is extended to algebraic spaces. The functoriality of $X^+$ is proven.

The proof is based on the results of J. Alper et al. [Ann. Math. (2) 191, No. 3, 675–738 (2020; Zbl 1461.14017)] which says that in étale topology and over an algebraically closed field every fixed point has an affine $G$-invariant neighbourhood. The result in the affine situation is obtained by looking at the formal neighbourhood of $X^G$.

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