Banakh, Taras; Bardyla, Serhii; Ravsky, Alex
Embedding topological spaces into Hausdorff \(\kappa\)-bounded spaces. (English) [Zbl 1448.54013]

The system ZFC is the set-theoretic framework for this article. Let \(\kappa\) be an infinite cardinal and let \(X\) be a topological space. Then \(X\) is called \(\kappa\)-bounded if, for every subset \(A\) of \(X\) such that \(|A| \leq \kappa\), the closure of \(A\) in \(X\) is compact. For a non-empty family \(\mathcal{F}\) of closed subsets of \(X\), the authors introduce the notions of an \(\mathcal{F}\)-regular, strongly \(\mathcal{F}\)-regular, \(\mathcal{F}\)-Tychonoff, \(\mathcal{F}\)-normal, strongly \(\mathcal{F}\)-normal and a totally \(\mathcal{F}\)-normal space. For the family \(\mathcal{F}\) of closed subsets of the closures of subsets of cardinality \(\leq \kappa\) of \(X\), the space \(X\) is called \(\pi\)-regular (respectively, strongly \(\pi\)-regular and so on) if \(X\) is \(\mathcal{F}\)-regular (respectively, strongly \(\mathcal{F}\)-regular and so on). For the Wallman extension \(WX\) of a \(T_1\)-space \(X\), let \(j_X\) be the canonical embedding of \(X\) into \(WX\). Then the subspace \(W\mathcal{F}X = \bigcup \{j_X(C) : C \subseteq X \land |C| \leq \kappa\}\) of \(WX\) is called the Wallman \(\kappa\)-bounded extension of \(X\).

For a \(T_1\)-space \(X\) and an arbitrary infinite cardinal \(\kappa\), the authors consider the following statements: (1) \(X\) is \(\pi\)-normal, (2) \(W\pi X\) is Hausdorff, (3) \(X\) is homeomorphic to a subspace of a Hausdorff \(\kappa\)-bounded space, (4) \(X\) is \(\pi\)-regular, (5) \(X\) is strongly \(\pi\)-normal, (6) \(W\pi X\) is Urysohn, (7) \(X\) is homeomorphic to a subspace of a Urysohn \(\kappa\)-bounded space; (8) \(X\) is strongly \(\pi\)-regular, (9) \(X\) is totally \(\pi\)-normal, (10) \(W\pi X\) is regular, (11) \(X\) is homeomorphic to a subspace of a regular \(\kappa\)-bounded space, (12) \(X\) is regular. The authors show that (1) \(\leftrightarrow\) (2) \(\rightarrow\) (3) \(\leftrightarrow\) (4), (5) \(\leftrightarrow\) (6) \(\rightarrow\) (7) \(\rightarrow\) (8) and (9) \(\leftrightarrow\) (10) \(\rightarrow\) (11) \(\rightarrow\) (12).

Furthermore: if each closed subspace of \(X\) of density \(\leq \kappa\) is Lindelöf, then (4) \(\rightarrow\) (1); if each closed subspace of \(X\) of density \(\leq \kappa\) is countably paracompact in \(X\) and Lindelöf, then (8) \(\rightarrow\) (5); if each closed subspace of \(X\) of density \(\leq \kappa\) is paracompact in \(X\), then (12) \(\rightarrow\) (9).

By giving suitable examples, the authors show that the following spaces exist in ZFC: (a) a first-countable \(\omega\)-normal \(T_3\)-space which is neither functionally Hausdorff nor strongly \(\omega\)-normal; (b) a Tychonoff, zero-dimensional, locally compact, locally countable \(\omega\)-normal space which is not strongly \(\omega\)-normal; (c) a totally \(\omega\)-normal, \(\omega\)-bounded \(T_3\)-space which is not functionally Hausdorff; (d) a \(\pi\)-normal, \(\kappa\)-bounded, \(H\)-compact Hausdorff space which is not Urysohn.

Finally, the authors prove that it is consistent with ZFC that there exists a separable, sequentially compact scattered \(T_1\)-space which is not \(\pi\)-regular, so it cannot be embedded into an \(\omega\)-bounded Hausdorff space. The problem of whether there exists in ZFC a separable, sequentially compact \(T_1\)-space which is not Tychonoff is posed.

Reviewer: Eliza Wajch (Siedlce)

MSC:
- 54D30 Compactness
- 54B35 Extensions of spaces (compactifications, supercompactifications, completions, etc.)
- 54D80 Special constructions of topological spaces (spaces of ultrafilters, etc.)
- 54B30 Categorical methods in general topology

Keywords:
- \(\kappa\)-bounded space; Wallman extension; countably compact space; sequentially compact space

Full Text: [DOI]

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