In this extremely well-written and deep article, the authors study quadric rank loci on moduli of curves and K3 surfaces. Let $X$ be an algebraic variety (over $\mathbb{C}$) and let $\mathcal{E}, \mathcal{F}$ be two vector bundles on $X$ having ranks $e$ and $f$, respectively. Assume that we have the following morphism of vector bundles

$$\phi : \text{Sym}^2(\mathcal{E}) \to \mathcal{F}.$$ 

For a positive integer $r \leq e$ we define the subvariety of $X$ consisting of all points for which $\text{Ker}(\phi)$ contains a quadric of corank at least $r$, that is

$$\Sigma_{e,f}^r := \{ x \in X : 0 \neq q \in \text{Ker}(\phi(x)) \text{ with } \text{rk}(q) \leq e - r \}.$$ 

Since the codimension of the variety of symmetric $e \times e$ matrices of corank $r$ is equal to $\binom{e+1}{2} - \binom{r+1}{2}$, it follows that the expected codimension of the locus $\Sigma_{e,f}^r(\phi)$ is equal to $\binom{e+1}{2} - \binom{r+1}{2} + f + 1$. The main aim of the paper under review is to determine explicitly the cohomology class of this locus in terms of the Chern classes of $\mathcal{E}$ and $\mathcal{F}$. This goal was achieved for every $e, f$ and $r$ in Theorem 4.4 therein (which is quite technical so that’s why we are not going to recall it here). However, if this locus is expected to be a divisor, then the formula has a very nice and simpler form.

Theorem A. We fix integers $0 \leq r \leq e$ and set $f := \binom{e+1}{2} - \binom{r+1}{2}$. Suppose that $\phi : \text{Sym}^2(\mathcal{E}) \to \mathcal{F}$ is a morphism of vector bundles over $X$. The class of the virtual divisor $\Sigma_{e,f}^r$ is given by the formula

$$[\Sigma_{e,f}^r(\phi)] = A_r^e \left( c_1(\mathcal{F}) - \frac{2f}{e} c_1(\mathcal{E}) \right) \in H^2(X, \mathbb{Q}),$$

where $A_r^e = \binom{e}{r}(\binom{e+1}{2} - \binom{r+1}{2})^{-1}$. It is worth noticing that $A_r^e$ is the degree of the variety of symmetric $e \times e$ matrices of corank at least $r$ inside the projective space of all symmetric $e \times e$ matrices.

In order to formulate the second crucial result of the paper under review, we need the following definition. If $V$ is a vector space, a pencil of quadrics $\ell \subset \mathbb{P}(\text{Sym}^2 V)$ is said to be degenerate if the intersection of $\ell$ with the discriminant divisor $D(V) \subset \mathbb{P}(\text{Sym}^2(V))$ is non-reduced. Now we consider a morphism $\phi : \text{Sym}^2(\mathcal{E}) \to \mathcal{F}$ such that all kernels are expected to be pencils of quadrics and we impose the condition that the pencil is degenerate.

Theorem B. We fix integers $e$ and $f = \binom{e+1}{2} - 2$ and let $\phi : \text{Sym}^2(\mathcal{E}) \to \mathcal{F}$ be a morphism of vector bundles. The class of the virtual divisor $\mathfrak{D}_p := \{ x \in X : \text{Ker}(\phi(x)) \text{ is a degenerate pencil} \}$ equals to

$$[\mathfrak{D}_p] = (e - 1) \left( ec_1(\mathcal{F}) - (e^2 + e - 4)c_1(\mathcal{E}) \right) \in H^2(X, \mathbb{Q}).$$

The above results are motivated by a number of important questions in moduli theory and what follows the authors deliver a plethora of applications of these results in various situations, namely:

i) they find a simple proof of Borcherds’ result that the Hodge class on the moduli space of polarized K3 surfaces of fixed genus is of Noether-Lefschetz type;

ii) they construct an explicit canonical divisor on the Hurwitz space parametrizing degree $k$ covers of $\mathbb{P}^1$ from curves of genus $2k - 1$;

iii) they provide a closed formula for the Petri divisor on $\overline{M}_g$ of canonical curves which lie on a rank 3 quadrics;

iv) and finally they construct myriads of effective divisors of small slope on $\overline{M}_g$.

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MSC:
14C17 Intersection theory, characteristic classes, intersection multiplicities in algebraic geometry
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