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This paper gives some asymptotic counting formulas for rational points of bounded height on anisotropic tori, in the following setting:

Let \( F \) be a number field, and \( T \) be a torus whose scheme of characters has no rational points, beside the trivial character. Let \( X \) be toric variety with respect to \( T \), and write \( U \subset X \) for the dense open orbit, which is a principal homogeneous space for the torus. Furthermore, let \( \mathcal{B} \subset \text{Br}(U) \) be a finite subgroup consisting of Brauer classes that vanish after base-change to the algebraic closure \( F^{\text{alg}} \). Let \( U(F)_{\mathcal{B}} \) be the ensuing set of rational points over which all members of \( \mathcal{B} \) become trivial, and assume that this set in non-empty. The main result asserts that there is a constant \( c > 0 \) with

\[
N(U, H, \mathcal{B}, B) \sim c B \left( \frac{\log B}{\log B} \right)^{\rho - 1} \Delta, \quad \text{as } B \to \infty.
\]

The left-hand side \( N(U, H, \mathcal{B}, B) \) counts the number of rational points \( x \in U(F)_{\mathcal{B}} \) with bounded height \( H(x) \leq B \). Here \( \rho = \rho(X) \) is the Picard number of the toric variety \( X \), and

\[
\Delta = \sum_{D \in X^{(1)}} \left( 1 - \frac{1}{|\partial_D(\mathcal{B})|} \right)
\]

is a rational number measuring the size of the finite group \( \mathcal{B} \) under the residue maps at codimension-one points \( D \in X \), and \( H \) is the Batyrev-Tschinkel anticanonical height function [V. V. Batyrev and Y. Tschinkel, Int. Math. Res. Not. 1995, No. 12, 591–635 (1995; Zbl 0890.14008)]. It follows that the non-zero set \( U(F)_{\mathcal{B}} \) is infinite, and it is actually shown to be Zariski dense.

The author also gives an interpretation of the leading constant \( c = c_{X, \mathcal{B}, H} \) in terms of Artin L-functions, Tamagawa numbers, and Picard groups. This formally resembles the leading constant \( c = c_{X, H, \text{Peyre}} \) conjectured to appear in the context of Manin’s Conjecture [E. Peyre, Duke Math. J. 79, No. 1, 101–218 (1995; Zbl 0901.14025)].

From the above result, a similar asymptotic formula for certain families \( \pi : Y \to X \) is derived, where the restriction to \( U \) becomes a product of Brauer-Severi varieties. Now one counts points on \( U \), as above, but only those that lie in the image of \( Y(F) \).

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