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Long-time asymptotics of non-degenerate non-linear diffusion equations. (English)

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Summary: We study the long-time asymptotics of prototypical non-linear diffusion equations. Specifically, we consider the case of a non-degenerate diffusivity function that is a (non-negative) polynomial of the dependent variable of the problem. We motivate these types of equations using Einstein's random walk paradigm, leading to a partial differential equation in non-divergence form. On the other hand, using conservation principles leads to a partial differential equation in divergence form. A transformation is derived to handle both cases. Then, a maximum principle (on both an unbounded and a bounded domain) is proved in order to obtain bounds above and below for the time-evolution of the solutions to the non-linear diffusion problem. Specifically, these bounds are based on the fundamental solution of the linear problem (the so-called Aronson's Green function). Having thus sandwiched the long-time asymptotics of solutions to the non-linear problems between two fundamental solutions of the linear problem, we prove that, unlike the case of degenerate diffusion, a non-degenerate diffusion equation's solution converges onto the linear diffusion solution at long times. Select numerical examples support the mathematical theorems and illustrate the convergence process. Our results have implications on how to interpret asymptotic scalings of potentially anomalous diffusion processes (such as in the flow of particulate materials) that have been discussed in the applied physics literature.

MSC:

35B40 Asymptotic behavior of solutions to PDEs
35K59 Quasilinear parabolic equations
35K15 Initial value problems for second-order parabolic equations
35B50 Maximum principles in context of PDEs

Keywords:

Einstein's random walk paradigm; Aronson's Green function; anomalous diffusion processes

Software:

Matlab; ode23s; Ode15s; ode113; MATLAB ODE suite; ode45; ode23

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