Let $P_{n,k}$ be a principal $U(n)$-bundle over $S^2$ such that $c_1(P_{n,k}) = k \in \mathbb{Z}$. Then the gauge group $G(P_{n,k})$ is the topological group of $U(n)$-equivariant automorphisms of $P_{n,k}$ fixing $S^2$. According to [M. F. Atiyah and R. Bott, Philos. Trans. R. Soc. Lond., Ser. A 308, 523–615 (1983; Zbl 0509.14014)] and [D. H. Gottlieb, Trans. Am. Math. Soc. 171, 23–50 (1972; Zbl 0251.55018)], the classifying space of $G(P_{n,k})$ is homotopy equivalent to $\text{Map}(S^2, BU(n); k)$, the connected component of $\text{Map}(S^2, BU(n))$ that contains the the degree $k$ map $S^2 \to BU(n)$. In this paper the author calculates the cohomology ring $H^*(BG(P_{n,k})) \cong H^*(\text{Map}(S^2, BU(n); k))$.

In Section 2, the author introduces the free double suspension $\hat\sigma^2_k : H^n(X) \to H^{n-2}(\text{Map}(S^2, X;k))$, which generalizes the suspension operation $\sigma : H^n(X) \to H^{n-1}(\Omega X)$. Let $\phi$ be the composition
$$\phi : \text{Map}(S^2, BU(n); k) \to \text{Map}(S^2, BU(\infty); k) \to \Omega^2_k BU(\infty) \to \Omega^3_0 BU(\infty) \to BU(\infty)$$
where the first map is induced by inclusion $BU(n) \to BU(\infty)$, the second map is a retraction, the third map is induced by the concatenation with the degree $-k$ map $S^2 \to BU(\infty)$, and the last map is the Bott periodicity homotopy equivalence. Let $x_i = \phi^*(c_i)$ be the image of the $i$th Chern class of the universal bundle. In Section 3, the author applies the Leray-Hirsch spectral sequence to the evaluation fibration $\Omega^2_k(BU(n)) \to \text{Map}(S^2, BU(n); k) \xrightarrow{\psi} BU(n)$ and uses properties of $\hat{\sigma}^2_k$ to show that there is a ring isomorphism
$$\mathbb{Z}[c_1, \ldots, c_n, x_1, x_2, \ldots]/(h_n, h_{n+1}, \ldots) \to H^*(\text{Map}(S^2, BU(n); k)),$$
where $h_i = kc_i + \sum_{1 \leq j \leq i} (-1)^j s_j(x_1, \ldots, x_j)c_{i-j}$ and $s_j(\sigma_1, \ldots, \sigma_j) = \sum_{i=1}^n t_i^j$ is the $j$th Newton polynomial. Moreover, he constructs a virtual bundle $\zeta \in K(BG(P_{n,k}))$ such that $c_i(\zeta) = x_i$.

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