**Lamy, Stéphane**; **Zimmermann, Susanna**

**Signature morphisms from the Cremona group over a non-closed field.** (English)

Zbl 1456.14019
J. Eur. Math. Soc. (JEMS) 22, No. 10, 3133-3173 (2020).

The Cremona group $\mathrm{Cr}_2(\Bbbk)$ is the group of birational transformations of the projective plane $\mathbb{P}^2$ over a field $\Bbbk$. It a long-standing question whether $\mathrm{Cr}_n(\mathbb{C})$ is simple group. Several years ago *S. Cantat* et al. made a breakthrough in [Acta Math. 210, No. 1, 31–94 (2013; Zbl 1278.14017)] by proving that $\mathrm{Cr}_2(\mathbb{C})$ is not simple. *A. Lonjou* generalized the result to an arbitrary field in [Ann. Inst. Fourier 66, No. 5, 2021–2046 (2016; Zbl 1365.14017)]. Over the field of complex numbers it was classically known that $\mathrm{Cr}_2(\Bbbk)$ does not admit any non trivial homomorphism to an abelian group. Over the field of real numbers *S. Zimmermann* proved in [Duke Math. J. 167, No. 2, 211–267 (2018; Zbl 1402.14015) ] that the abelianization of $\mathrm{Cr}_2(\mathbb{R})$ is a direct sum of uncountably many $\mathbb{Z}/2\mathbb{Z}$.

The article under review deals with an arbitrary perfect field with at least one Galois extension of degree eight. The authors constructed a tree on which $\mathrm{Cr}_2(\Bbbk)$ acts so that $\mathrm{Cr}_2(\Bbbk)$ can be written as an amalgam product by Bass-Serre theory. Note that each factor in the amalgam product is a big group and there are a lot of factors (same cardinality as the field $\Bbbk$). Consequently the authors constructed a homomorphism from $\mathrm{Cr}_2(\Bbbk)$ to a free product of $\mathbb{Z}/2\mathbb{Z}$, thus also a homomorphism from $\mathrm{Cr}_2(\Bbbk)$ to a direct sum of $\mathbb{Z}/2\mathbb{Z}$.

The tree mentionned above comes from a square complex constructed in this paper on which $\mathrm{Cr}_2(\Bbbk)$ acts. The vertices of the square are rank $r$ fibrations with $r = 1, 2, 3$; rank $r$ fibrations are generalizations of Mori fiber spaces. Roughly speaking the edges and the faces of the square complex record Sarkisov links and relations among Sarkisov links. If we blow up a general point of degree eight on $\mathbb{P}^2$ then we obtain a del Pezzo surface of degree 1. Such a del Pezzo surface gives a rank 2 fibration and an element in $\mathrm{Cr}_2(\Bbbk)$ called a Bertini involution. This is where the hypothesis on the field $\Bbbk$ is used. Rougly speaking the tree is constructed by recording the action of $\mathrm{Cr}_2(\Bbbk)$ on the part of the square complex containing these Bertini involutions.

Reviewer: Shengyuan Zhao (Stony Brook)

**MSC:**

| | | |
|---|---|---|
| 14E07 | Birational automorphisms, Cremona group and generalizations | Cited in **1** Review |
| 14E30 | Minimal model program (Mori theory, extremal rays) | Cited in **3** Documents |

**Keywords:**

Cremona group; Sarkisov program; amalgam product

**Full Text:** DOI arXiv

**References:**

[1] Bauer, T., Küronya, A., Szemberg, T.: Zariski chambers, volumes, and stable base loci. J. Reine Angew. Math.576, 209-233 (2004)Zbl 1055.14007 MR 2099205 · Zbl 1055.14007

[2] Cantat, S.: The Cremona group. In: Algebraic Geometry: Salt Lake City 2015, Proc. Sympos. Pure Math. 97.1, Amer. Math. Soc., 101-142 (2017)MR 3821147 · Zbl 1451.14037

[3] Cantat, S., Lamy, S.: Normal subgroups in the Cremona group (with an appendix by Y. de Cornulier). Acta Math.210, 31-94 (2013)Zbl 1278.14017 MR 3037611 · Zbl 1278.14017

[4] Dean, R. A.: A rational polynomial whose group is the quaternions. Amer. Math. Monthly88, 42-45 (1981)Zbl 0458.12013 MR 0619418 · Zbl 0458.12013

[5] Debarre, O.: Higher-Dimensional Algebraic Geometry. Universitext, Springer, New York (2001)Zbl 0978.14001 MR 1841091 · Zbl 0978.14001

[6] Hacon, C. D., McKernan, J.: The Sarkisov program. J. Algebraic Geom.22, 389-405 (2013)Zbl 1267.14024 MR 3019454 · Zbl 1267.14024

[7] Hatcher, A.: Algebraic Topology. Cambridge Univ. Press, Cambridge (2002) Zbl 1044.55001 MR 1867354 · Zbl 1044.55001

[8] Iskovskikh, V. A.: Generators in the two-dimensional Cremona group over a nonclosed field. Trudy Mat. Inst. Steklova200, 157-170 (1991) (in Russian); English transl.: Proc. Steklov Inst. Math.200, 173-188 (1993)Zbl 0811.14010 MR 1143365 · Zbl 0811.14010

[9] Iskovskikh, V. A.: Factorization of birational mappings of rational surfaces from the point of view of Mori theory. Uspekhi Mat. Nauk51, no. 4, 3-72 (1996) (in Russian) Zbl 0914.14005 MR 1422227

[10] Iskovskikh, V. A., Kabdykairov, F. K., Tregub, S. L.: Relations in a two-dimensional Cremona group over a perfect field. Izv. Ross. Akad. Nauk Ser. Mat.57, 3-69 (1993) (in Russian); English transl.: Russian Acad. Sci. Izv. Math.42, 427-478 (1994) Zbl 0824.14010 MR 1243341 · Zbl 0824.14010

[11] Kaloghiros, A.-S.: Relations in the Sarkisov program. Compos. Math.149, 1685-1709 (2013)Zbl 1280.14004 · Zbl 1280.14004

[12] Kaloghiros, A.-S., Küronya, A., Lazi´c, V.: Finite generation and geography of models. In: Minimal Models and Extremal Rays, Adv. Stud. Pure Math. 70, Math. Soc. Japan, Tokyo, 215-245 (2016)Zbl 1369.14025 MR 3617781 · Zbl 1369.14025

[13] Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties. Cambridge Tracts in Math. 134, Cambridge Univ. Press, Cambridge (1998)Zbl 1143.14014 MR 1658959

[14] Liedtke, C.: Algebraic surfaces in positive characteristic. In: Birational Geometry, Rational Curves, and Arithmetic, Springer, New York, 229-292 (2013)Zbl 1312.14001 MR 3114931 · Zbl 1312.14001

[15] Liu, Q.: Algebraic Geometry and Arithmetic Curves. Oxford Grad. Texts in Math. 6, Oxford Univ. Press, Oxford (2002)Zbl 1103.14001 MR 1917232 · Zbl 0996.14005

[16] Lonjou, A.: Non simplicité du groupe de Cremona sur tout corps. Ann. Inst. Fourier (Grenoble)66, 2021-2046 (2016)Zbl 1365.14017 MR 3533276 · Zbl 1365.14017

[17] Matsuki, K.: Introduction to the Mori Program. Universitext. Springer, New York (2002) Zbl 0988.14007 MR 1875410 · Zbl 0988.14007

[18] Morandi, P.: Field and Galois Theory. Grad. Texts in Math. 167, Springer, New York (1996)Zbl 0865.12001 MR 1410264

[19] Munkres, J. R.: Elements of Algebraic Topology. Addison-Wesley, Menlo Park, CA (1984)Zbl 0673.55001 MR 0755006 · Zbl 0673.55001

[20] Reid, M.: Chapters on algebraic surfaces. In: Complex Algebraic Geometry (Park City, UT, 1993), IAS/Park City Math. Ser. 3, Amer. Math. Soc., Providence, RI, 3-159 (1997) Zbl 0910.14016 MR 1442522 · Zbl 0910.14016

[21] Rosser, J. B., Schoenfeld, L.: Approximate formulas for some functions of prime numbers. Illinois J. Math.6, 64-94 (1962)Zbl 0122.05001 MR 0137689 · Zbl 0122.05001

[22] Sansuc, J.-J.: Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres. J. Reine Angew. Math.327, 12-80 (1981)Zbl 0468.14007 MR 0631309 · Zbl 0468.14007

[23] Serre, J.-P.: Trees. Springer Monogr. Math., Springer, Berlin (2003)Zbl 1013.20001 MR 1954121

[24] Silverman, J. H.: The Arithmetic of Elliptic Curves. Grad. Texts in Math. 106, Springer, Dordrecht (2009)Zbl 1194.11005 MR 2514094

[25] Terakawa, H.: Thed-very ampleness on a projective surface in positive characteristic. Pacific J. Math.187, 187-199 (1999)Zbl 0967.14008 MR 1674325 · Zbl 0967.14008

[26] Zimmermann, S.: The real Cremona group is a nontrivial amalgam. Ann. Inst. Fourier (Grenoble), to appear;arXiv:1705.07500(2017)

[27] Zimmermann, S.