Given a unital quantale \((Q, \otimes, e)\), there exists the category \(\text{Mod}_Q\) of right unital \(Q\)-modules, whose objects are pairs \((M, \Box)\), where \(M\) is a complete lattice, and \(\Box : M \times Q \to M\) is a map with the following four properties: (1) \(\forall S \subseteq M\) and every \(q \in Q\); (2) \(m \Box (\vee T) = \bigvee_{t \in T} (m \Box t)\) for every \(m \in M\) and every \(T \subseteq Q\); (3) \(m \Box q_1 \Box q_2 = m \Box (q_1 \otimes q_2)\) for every \(m \in M\) and every \(q_1, q_2 \in Q\); and (4) \(m \Box e = m\) for every \(m \in M\) (see, [D. Kruml and J. Paseka, Handb. Algebra 5, 323–362 (2008; Zbl 1219.06016)]) for more detail). In particular, if \(Q\) is the two-element quantale \(2 = \{\bot, \top\}\), then the category \(\text{Mod}_Q\) is isomorphic to the category \(\text{Sup}\) of complete lattices and join-preserving maps. If \(Q\) has additionally an involution \(\prime\), then there exists a self-duality on the category \(\text{Mod}_Q\), i.e., a contravariant functor \(S : \text{Mod}_Q \to \text{Mod}_Q\) such that \(S \circ S = 1_{\text{Mod}_Q}\), defined on an object \((M, \Box)\) by \(S(M, \Box) = (M^{\text{op}}, \Box^{\text{op}})\), where \(M^{\text{op}}\) is the dual complete lattice of \(M\), and \(m^{\text{op}} \Box q = \bigvee\{n \in M \mid n \Box q' \leq m\}\); and on a morphism \(f : (M, \Box) \to (N, \Box)\) by \(S(f) = f^{\text{op}}\), where \(f^{\text{op}} : N^{\text{op}} \to M^{\text{op}}\) is the right adjoint map of \(f\) (in the sense of partially ordered sets).

Based in their previous study of quantales and quantale modules in [P. Eklund et al., Semigroups in complete lattices. Quantales, modules and related topics. Cham: Springer (2018; Zbl 06863918)] and the paper of I. Stubbe [Theor. Comput. Sci. 373, No. 1–2, 142–160 (2007; Zbl 1111.68073)], who showed that projectivity in the category \(\text{Mod}_Q\) is equivalent to complete distributivity enriched over \(Q\), the present authors consider preservation of projectivity by the above-mentioned duality \(S\). More precisely, they show that if the underlying quantale \(Q\) is unital and involutive with a designated element (see Definition 4.5 for more detail), then the duality \(S\) preserves projectivity if and only if \(Q\) has a dualizing element (Theorem 4.11). The last section of the paper shows that involutive quantales from the above result are widespread, e.g., they are induced by completely distributive lattices with an order-reversing involution or by arbitrary groups.

The paper is well written, provides most of its required preliminaries, and will be of interest to all the researchers studying categories enriched in quantales.

Reviewer: Sergejs Solovjovs (Praha)

MSC:

- 18D20  Enriched categories (over closed or monoidal categories)
- 06F07  Quantales
- 03G10  Logical aspects of lattices and related structures
- 06D10  Complete distributivity
- 06D75  Other generalizations of distributive lattices
- 18B25  Topoi

Keywords:

- complete distributivity
- complete lattice
- cyclic element
- dualizing element
- enriched category
- monoidal category
- preordered set
- presheaf
- projective module
- quantale
- quantale module
- totally below relation

Full Text: DOI

References:
