This paper is concerned with the sum-product phenomenon which says that either the sum set $A + B = \{a + b : a \in A, b \in B\}$ or the product set $AB = \{ab : a \in A, b \in B\}$ must be large. It treats the case of sets in a finite field $\mathbb{F}_p$. The introduction contains a wide-ranging account of approaches to the sum-product phenomenon in $\mathbb{R}$ and $\mathbb{F}_p$, explaining the significance of incidence geometry, hyperbolas, energy and bilinear forms of Kloosterman sums with many references to results and records in the literature.

The following is an example of a new incidence theorem for a hyperbola. Let $A, B, C, D \subseteq \mathbb{F}_p$ be sets. For any $\lambda \geq 0$, the number of solutions $(a, b, c, d) \in A \times B \times C \times D$ on the hyperbola $(a + b)(c + d) = \lambda$ differs asymptotically from the expected value $|A||B||C||D|/p$ by at most $|A|^{1/4}|B||C||D|^{1/2} + |A|^{3/4}(|B||C|)^{1/8}|D|^{1/2}$.

The incidence theorems are connected with Kloosterman sums. The Kloosterman sum in $\mathbb{F}_p$ is $K(n, m) = \sum_{x \in \mathbb{F}_p} e(nx + mx^{-1})$ where $e(\cdot)$ is an additive character on $\mathbb{F}$. Consider the bilinear form $S(\alpha, \beta) = \sum_{n,m} \alpha(n)\beta(m)K(n,m)$ where $\alpha, \beta : \mathbb{F} \to \mathbb{C}$ are arbitrary functions. For the application, let $c > 1$, $t_1, t_2 \in \mathbb{F}_p$ and $N,M$ be integers with $N,M \leq p^{1-c}$ and let $\alpha, \beta : \mathbb{F} \to \mathbb{C}$ be functions supported on $\{1, \ldots, N\} + t_1$ and $\{1, \ldots, M\} + t_2$ respectively. Then there exists $\delta > 0$ depending on $c$ such that $S(\alpha, \beta) \ll |\alpha||\beta|^{2p^{-1-\delta}}$. More explicit bounds depending on $N, M$ and $p$ are obtained for certain choices of $\alpha$ and $\beta$.

The methods are combinatorial. The first part concerns the function $T_k(A_1, \ldots, A_{2k})$ which counts the number of solutions of $a_1a_2^{-1} \ldots a_{k-1}a_k^{-1} = a_{k+1}a_{k+2}^{-1} \ldots a_{2k-1}a_{2k}$ where $a_i \in A_i$ and the $A_i$ are sets in a general group. The hyperbola $(y - 1)(b - x) = \lambda$ can be written $y = gx = u_axv_bx$ where $u_a = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $v_b = \left(\begin{smallmatrix} a & b \\ 1 & 0 \end{smallmatrix}\right)$ and $g = u_av_b = \left(\begin{smallmatrix} ab+\lambda \\ b \end{smallmatrix}\right) \in G_\lambda(A,B)$. One connection with the energy is $T_2(G_\lambda(A,B)) \leq |A|^2E^-(B) + |B|^2E^+(A)$ and there are similar results for higher energies. Application of a sum-product result [I. D. Shkredov, Trans. Mosc. Math. Soc. 2018, 231–281 (2018; Zbl 07050908); translation from Tr. Mosk. Mat. O.-va 70, No. 2, 271–334 (2018)] and estimates for $T_2(G)$ and $T_3(G)$ leads to the incidence theorem stated earlier and other consequences such as a bound for the size of a hyperbola with elements from a set with small sum set. Variations on this theme derived from estimates for $T_{2k}$ lead to non-trivial incidence theorems in both $\mathbb{Z}$ and $\mathbb{F}_p$ for various ranges of the parameters. The machinery also leads to estimates for bilinear Kloosterman sums with explicit dependence on the parameters when the supports of the weights belong to arithmetic progressions.

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MSC:

11B30 Arithmetic combinatorics; higher degree uniformity
11T24 Other character sums and Gauss sums
11L05 Gauss and Kloosterman sums; generalizations

Keywords:

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