Let \((X, d, m)\) be an RCD\((K, \infty)\) space, that is a metric measure space satisfying the Riemannian curvature-dimension condition. Here \(K\) can be viewed as a lower bound on Ricci curvature and \(N = \infty\) as an upper bound for dimension. There are many important works on RCD \((K, \infty)\) and RCD \((K, N)\) for \(N < \infty\) in recent years. In particular, on an RCD \((K, \infty)\) space with \(K > 0\), there is a spectral gap \(\lambda_1 > K\) for the Laplacian operator \(-\Delta\). The main purpose of this paper is to obtain a rigidity theorem for the spectral gap.

The main result of this paper is stated in Theorem 1. Since the Laplacian has discrete spectrum \(\{\lambda_i\}_{i=0}^{\infty}\), assuming that \(\lambda_i = K\) for \(1 \leq i \leq k\) on an RCD \((K, \infty)\) space \((X, d, m)\) with \(K > 0\), the authors showed that \((X, d, m)\) splits off \(k\)-dimensional Gaussian space. Precisely, \((X, d)\) is isometric to the product of an RCD \((K, \infty)\) space \((Y, d_Y)\) with \((\mathbb{R}^k, |\cdot|)\) and the measure \(m\) coincides with the product measure \(m_Y\) with Gaussian measure \(e^{-K|x|^2/2}dx_1 \cdots dx^k\).

The above rigidity result was previously proved by X. Cheng and D. Zhou [Commun. Contemp. Math. 19, No. 1, Article ID 1650001, 17 p. (2017; Zbl 1360.58022)] for smooth metric measure space with \(\infty\)-Bakry-Émery Ricci curvature bounded from below by positive constant. The main idea is to use the Bochner formula together with the equality for the spectral gap to force the Hessian of the eigenfunction to be affine. In order to generalize the result to an RCD \((K, \infty)\) space \((X, d, m)\) with \(K > 0\), the argument by Cheng and Zhou can not be applied directly as the metric measure space is not smooth now, and the argument for the rigidity of spectral gap on RCD \((K, N)\) space with \(N < \infty\) does not work as well. To overcome the difficulties the authors consider the lift \(U\) of the eigenfunction \(u\) to the \(L^2\)-Wasserstein space and employ the theory of the regular Lagrangian flow that was developed in L. Ambrosio and D. Trevisan [Anal. PDE 7, No. 5, 1179–1234 (2014; Zbl 1357.49058)] recently.

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