

**Möller, Martin; Ulirsch, Martin; Werner, Annette**

**Realizability of tropical canonical divisors.** (English) Zbl 1466.14033  
J. Eur. Math. Soc. (JEMS) 23, No. 1, 185-217 (2021).

This paper studies the realizability problem for effective tropical canonical divisors in equicharacteristic zero. More precisely, given a stable tropical curve  $\Gamma$  and a divisor  $D$  in the canonical linear system on  $\Gamma$ , the authors provide a combinatorial condition to decide when there exists a smooth curve  $X$  over a non-Archimedean field whose stable reduction has  $\Gamma$  as its dual tropical curve together with an effective canonical divisor  $K_X$  that specializes to  $D$ . The proof of the result relies on the description of the incidence variety compactification of the strata of abelian differentials with prescribed orders of zeros by *M. Bainbridge* et al. [Duke Math. J. 167, No. 12, 2347–2416 (2018; [Zbl 1403.14058](#))], where the dual graph  $\Gamma$  is equipped with an enhanced level structure and certain residue conditions have to be satisfied. The authors translate these conditions into tropical geometry and solve the corresponding combinatorial problem. In particular, the result implies that the canonical divisor  $K_\Gamma$  on  $\Gamma$  is in general not realizable. The authors also develop a moduli-theoretic framework to understand specialization of divisors from algebraic to tropical curves as a natural toroidal tropicalization map in the sense of *D. Abramovich* et al. [Ann. Sci. Éc. Norm. Supér. (4) 48, No. 4, 765–809 (2015; [Zbl 1410.14049](#))].

Reviewer: [Dawei Chen \(Chestnut Hill\)](#)

**MSC:**

- [14H15](#) Families, moduli of curves (analytic)
- [14H10](#) Families, moduli of curves (algebraic)
- [32G15](#) Moduli of Riemann surfaces, Teichmüller theory (complex-analytic aspects in several variables)

Cited in **1** Review  
Cited in **6** Documents

**Keywords:**

tropical geometry; moduli spaces; Hodge bundle; abelian differentials; canonical divisors; flat surfaces; Berkovich spaces

**Full Text:** [DOI](#) [arXiv](#)

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