Gandini, Jacopo; **Maffei, Andrea**

The Bruhat order on Hermitian symmetric varieties and on abelian nilradicals. (English)

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Let G be a simple algebraic group over an algebraically closed group. Let P be a parabolic subgroup of G with abelian unipotent radical P^u, and let L be a Levi subgroup of P. Then G/L is a Hermitian symmetric space. Let B be a Borel subgroup of G contained in P. The Bruhat order on Hermitian symmetric varieties referred to in the title of the article refers to the partial order defined by inclusions of B-orbit closures in G/L. It is named in analogy with the classical Bruhat order, defined by inclusions of Schubert varieties in G/B. Such Schubert varieties are parametrized by elements of the Weyl group of G thanks to the Bruhat decomposition, and the induced partial order on the Weyl group can be encoded in a fully combinatorial way.

The B-orbits in G/L have already been studied, notably by *R. W. Richardson* and *T. A. Springer* [Geom. Dedicata 35, No. 1–3, 389–436 (1990; Zbl 0704.20039)]. They are parametrized by combinatorial data as well. The main result of the paper is a combinatorial translation of the partial order alluded to above. This provides a solution to a conjecture of *R. W. Richardson* and *T. A. Springer* [Contemp. Math. 153, 109–142 (1993; Zbl 0840.20039),Conjecture 5.6.2].

One ingredient of the proof, which leads to results of independent interest is the following. The inclusion $L \subset P$ provides a homogeneous fibration of G/L onto G/P whose fibers are isomorphic to P/L. Under natural identifications and exponential map, this fiber is isomorphic to the Lie algebra $\mathfrak{p}^u$ of P^u. Orbits under B behave well under this projection, allowing to approach the problem by studying B-orbit closures in G/P (a well-known variation on the classical Bruhat order) and B-orbit closures in the fibers $\mathfrak{p}^u$. The latter is studied as a first step in the present article, providing in particular the final steps to settle a conjecture of *D. Panyushev* [Transform. Groups 22, No. 2, 503–524 (2017; Zbl 1377.22016), Conjecture 6.2].

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MSC:

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14M15 Grassmannians, Schubert varieties, flag manifolds

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