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On polynomially integrable Birkhoff billiards on surfaces of constant curvature. (English)
Zbl 1466.37049

J. Eur. Math. Soc. (JEMS) 23, No. 3, 995-1049 (2021).

Author's abstract: The polynomial version of the Birkhoff Conjecture on integrable billiards on complete simply connected surfaces of constant curvature (plane, sphere, hyperbolic plane) was first stated, studied and solved in a particular case by *S. V. Bolotin* [Mosc. Univ. Mech. Bull. 45, No. 2, 10–13 (1990; Zbl 0727.58025); translation from Vestn. Mosk. Univ., Ser. I 1990, No. 2, 33–36 (1990); Math. Notes 51, No. 2, 1 (1992; Zbl 0795.58028); translation from Mat. Zametki 51, No. 2, 20–28 (1992)] in 1990–1992. Here we present a complete solution of the polynomial version of the Birkhoff Conjecture. Namely, we show that every polynomially integrable real bounded planar billiard with C^2-smooth connected boundary is an ellipse. We extend this result to billiards with piecewise smooth and not necessarily convex boundary on an arbitrary two-dimensional simply connected complete surface of constant curvature: plane, sphere, Lobachevsky-Poincaré (hyperbolic) plane; each of them being modeled as a plane or a (pseudo-) sphere in $\mathbb{R}^3$ equipped with an appropriate quadratic form. Namely, we show that a billiard is polynomially integrable if and only if its boundary is a union of confocal conical arcs and appropriate geodesic segments. We also present a complexification of these results. These are joint results of *M. Bialy* and *A. E. Mironov* [J. Geom. Phys. 115, 150–156 (2017; Zbl 1375.37113); Philos. Trans. R. Soc. Lond., A, Math. Phys. Eng. Sci. 376, No. 2131, Article ID 20170418, 19 p. (2018; Zbl 1407.37057); J. Phys. A, Math. Theor. 49, No. 45, Article ID 455101, 18 p. (2016; Zbl 1353.37075); Philos. Trans. R. Soc. Lond., A, Math. Phys. Eng. Sci. 376, No. 2131, Article ID 20170418, 19 p. (2018; Zbl 1407.37057)] and the author [Mosc. Math. J. 14, No. 2, 239–289 (2014; Zbl 1334.37018); Dokl. Math. 98, No. 1, 382–385 (2018; Zbl 1400.37063); translation from Dokl. Akad. Nauk, Ross. Akad. Nauk 481, No. 6, 594–598 (2018)]. The proof is split into two parts. The first part is given in two papers by *M. Bialy* and *A. E. Mironov* (in Euclidean and non-Euclidean cases respectively). Their geometric construction reduced the Polynomial Birkhoff Conjecture to a purely algebro-geometric problem to show that an irreducible algebraic curve in $\mathbb{CP}^2$ with certain proper-ties is a conic. They have shown that its singular and inflection points lie in the complex light conic of the above-mentioned quadratic form. In the present paper we solve the above algebro-geometric problem completely.

Reviewer: Nicolai K. Smolentsev (Kemerovo)

MSC:

37J35 Completely integrable finite-dimensional Hamiltonian systems, integration methods, integrability tests
37C83 Dynamical systems with singularities (billiards, etc.)
37J39 Relations of finite-dimensional Hamiltonian and Lagrangian systems with topology, geometry and differential geometry (symplectic geometry, Poisson geometry, etc.)
53A10 Minimal surfaces in differential geometry, surfaces with prescribed mean curvature
14H20 Singularities of curves, local rings

Keywords:
billiard; geodesic billiard flow; polynomial integral; algebraic Birkhoff conjecture; surface of constant curvature; singularities of algebraic curves

Full Text: DOI arXiv

References:

[1] Abdrakhmanov, A. M.: Integrable billiards. Moscow Univ. Math. Bull.45, no. 6, 13-17 (1990) Zbl 0850.70144 MR 1095993 · Zbl 0850.70144

[2] Abdrakhmanov, A. M.: On integrable systems with elastic reflections. Moscow Univ. Math. Bull.45, no. 5, 14-16 (1990)Zbl 0850.70143 MR 1085235 · Zbl 0850.70143

[3] Advis-Gaete, L., Carry, B., Gualtieri, M., Guthmann, C., Reffet, E., Tokieda, T.: Golfer's dilemma. Amer. J. Phys.74, 497-501 (2006)

[4] Amiran, E.: Caustics and evolutes for convex planar domains. J. Differential Geometry28, 345-357 (1988)Zbl 0636.58033 MR 0961519 · Zbl 0636.58033

[5] Avila, A., De Simoi, J., Kaloshin, V.: An integrable deformation of an ellipse of small eccentricity is an ellipse. Ann. of Math. (2)184, 527-558 (2016)Zbl 1379.37104 MR 3548532 · Zbl 1379.37104

[6] Berger, M.: Seules les quadriques admettent des caustiques. Bull. Soc. Math. France123, 107-116 (1995)Zbl 0830.51009 MR 1330789 · Zbl 0830.51009

[7] Bialy, M.: Convex billiards and a theorem by E. Hopf. Math. Z.214, 147-154 (1993) Zbl 0790.58023 MR 1234604 · Zbl 0790.58023

[8] Bialy, M.: Hopf rigidity for convex billiards on the hemisphere and hyperbolic plane. Discrete Contin. Dynam. Systems33, 3903-3913 (2013)Zbl 1306.37059 MR 3038045 · Zbl 1306.37059

[9] Bialy, M.: On totally integrable magnetic billiards on constant curvature surface. Electron. Res. Announc. Math. Sci.19, 112-119 (2012)Zbl 1257.37027 MR 2999056 · Zbl 1257.37027

[10] Bialy, M., Mironov, A. E.: Angular billiard and algebraic Birkhoff conjecture. Adv. Math. 313, 102-126 (2017)Zbl 1364.37124 MR 3649222 · Zbl 1364.37124

[11] Bialy, M., Mironov, A. E.: Algebraic Birkhoff conjecture for billiards on sphere and hyperbolic plane. J. Geom. Phys.115, 150-156 (2017)Zbl 1375.37113 MR 3623621 · Zbl 1375.37113

[12] Bialy, M., Mironov, A. E.: On fourth-degree polynomial integrals of the Birkhoff billiard. Proc. Steklov Inst. Math.295, 27-32 (2016)Zbl 1371.37067 MR 3628512 · Zbl 1371.37067

[13] Bialy, M., Mironov, A. E.: Algebraic non-integrability of magnetic billiards. J. Phys. A49, no. 45, art. 455101, 18 pp. (2016)Zbl 1353.37075 MR 3568603 · Zbl 1353.37075

[14] Bialy, M., Mironov, A. E.: A survey on polynomial in momenta integrals for billiard problems. Philos. Trans. Roy Soc. A.376, no. 2131, art. 20170418, 19 pp. (2018)Zbl 1407.37057 MR 3868418 · Zbl 1407.37057

[15] Bolotin, S. V.: First integrals of systems with gyroscopic forces. Vestnik Moskov. Univ. Ser. I Mat. Mekh.1984, no. 6, 75-82, 113 (in Russian)Zbl 0597.70019 MR 0775310 · Zbl 0597.70019

[16] Bolotin, S. V.: Integrable Birkhoff billiards. Moscow. Univ. Math. Mech. Bull.45, no. 2, 10- 13 (1990)Zbl 0727.58025 MR 1064916 · Zbl 0727.58025

[17] Bolotin, S. V.: Integrable billiards on surfaces of constant curvature. Math. Notes51, 117-123 (1992)Zbl 0795.58028 MR 1165461 · Zbl 0795.58028

[18] Brieskorn,E.,Kn¨orrer,H.:PlaneAlgebraicCurves.Birkh¨auser,Basel(1986) Zbl 0588.14019 MR 0886476

[19] Delshams, A., Ram´ırez-Ros, R.: On Birkoff's [Birkhoff's] conjecture about convex billiards. In: Proceedings of the 2nd Catalan Days on Applied Mathematics (Odeillo, 1995), Collect. Etudes, Presses Univ. Perpignan, Perpignan, 85-94 (1995)´Zbl 0911.58021 MR 1609592 · Zbl 0911.58021

[20] Dragovi´c, V., Radnovi´c, M.: Integrable billiards and quadrics. Russian Math. Surveys65, 319-379 (2010)Zbl 1202.37078 MR 2668802

[21] Dragovi´c, V., Radnovi´c, M.: Bicentennial of the great Poncelet theorem (1813-2013): current advances. Bull. Amer. Math. Soc. (N.S.)51, 373-445 (2014)Zbl 1417.37034 MR 3196793

[22] Dragovi´c, V., Radnovi´c, M.: Pseudo-integrable billiards and arithmetic dynamics. J. Modern Dynam.8, 109-132 (2014)Zbl 1351.37160 MR 3296569

[23] Dragovi´c, V., Radnovi´c, M.: Periods of pseudo-integrable billiards. Arnold Math. J.1, 69-73 (2015)Zbl 1371.37070 MR 3331968

[24] Dragovi´c, V., Radnovi´c, M.: Pseudo-integrable billiards and double reflection nets. Russian Math. Surveys70, 1-31 (2015)Zbl 1358.37070 MR 3353115

[25] Glutsyuk, A.: On quadrilateral orbits in complex algebraic planar billiards. Moscow Math. J. 14, 239-289 (2014)Zbl 1334.37018 MR 3236494 · Zbl 1334.37018

[26] Glutsyuk, A. A.: On two-dimensional polynomially integrable billiards on surfaces of constant curvature. Doklady Math.98, 382-385 (2018)Zbl 1400.37063 · Zbl 1400.37063

[27] Glutsyuk, A., Shustin, E.: On polynomially integrable planar outer billiards and curves with symmetry property. Math. Ann.372, 1481-1501 (2018)Zbl 1402.37048 MR 3880305 · Zbl 1402.37048

[28] Greuel, G.-M., Lossen, C., Shustin, E.: Introduction to Singularities and Deformations. Springer, Berlin (2007)Zbl 1125.32013 MR 2290112 · Zbl 1125.32013

[29] Griffiths, P., Harris, J.: Principles of Algebraic Geometry. Pure Appl. Math. WileyInterscience, New York (1978)Zbl 0408.14001 MR 0507725 · Zbl 0408.14001

[30] Hironaka, H.: Arithmetic genera and effective genera of algebraic curves. Mem. Coll. Sci. Univ. Kyoto Sect.A30, 177-195 (1956)Zbl 0099.15702 MR 0090850 · Zbl 0099.15702

[31] Kaloshin, V., Sorrentino, A.: On local Birkhoff Conjecture for convex billiards. Ann. of Math. 188, 315-380 (2018)Zbl 1394.37093 MR 3815464 · Zbl 1394.37093

[32] Kozlov, V. V., Denisova, N. V.: Symmetries and topology of dynamical systems with two degrees of freedom. Russian Acad. Sci. Sb. Math.80, 105-124 (1995)Zbl 0818.70018 MR 1257339

[33] Kozlov, V. V., Treshchev, D. V.: Billiards: A Genetic Introduction to the Dynamics of Systems with Impacts. Math. Monogr. 89, Amer. Math. Soc., Providence, RI (1991)Zbl 0729.34027 MR 1118378 · Zbl 0751.70009

[34] Lazutkin, V. F.: The existence of caustics for a billiard problem in a convex domain. Math. USSR Izv.37, 186-216 (1973)Zbl 0256.52001 MR 0328219 · Zbl 0256.52001

[35] Marco, J.-P.: Entropy of billiard maps and a dynamical version of the Birkhoff conjecture. J. Geom. Phys.124, 413-420 (2018)Zbl 1388.37043 MR 3754521 · Zbl 1388.37043

[36] Milnor, J.: Singular Points of Complex Hypersurfaces. Princeton Univ. Press, Princeton (1968)Zbl 0184.48405 MR 0239612 · Zbl 0184.48405

[37] Poritsky, H.: The billiard ball problem on a table with a convex boundary—an illustrative dynamical problem. Ann. of Math. (2)51, 446-470 (1950)Zbl 0037.26802 MR 0032960 · Zbl 0037.26802

[38] Ramani, A., Kalliterakis, A., Grammaticos, B., Dorizzi, B.: Integrable curvilinear billiards. Phys. Lett. A115, 25-28 (1986)MR 0836094

[39] Shustin, E.: On invariants of singular points of algebraic curves. Math. Notes34, 962-963 (1983)Zbl 0549.14011 MR 0731335 · Zbl 0549.14011

[40] Tabachnikov, S.: Geometry and Billiards. Student Math. Library 30, Amer. Math. Soc., Providence, RI (2005)Zbl 1119.37001 MR 2168892

[41] Tabachnikov, S.: On algebraically integrable outer billiards. Pacific J. Math.235, 89-92 (2008)Zbl 1250.37022 MR 2379772 · Zbl 1250.37022

[42] Treschev, D.: Billiard map and rigid rotation. Phys. D.255, 31-34 (2013)Zbl 1417.37139 MR 3064868 · Zbl 1417.37139

[43] Treschev, D.: On a conjugacy problem in billiard dynamics. Proc. Steklov Inst. Math.289, 291-299 (2015)Zbl 1359.37089 MR 3486788 · Zbl 1359.37089

[44] Treschev, D.: A locally integrable multi-dimensional billiard system. Discrete Contin. Dynam. Systems37, 5271-5284 (2017)Zbl 1368.37063 MR 3668361 · Zbl 1368.37063

[45] Veselov, A. P.: Integrable systems with discrete time, and difference operators. Funct. Anal. Appl.22, 83-93 (1988)Zbl 0694.58020 MR 0947601 · Zbl 0694.58020

[46] Veselov, A. P.: Confocal surfaces and integrable billiards on the sphere and in the Lobachevsky space. J. Geom. Phys.7, 81-107 (1990)Zbl 0729.58029 MR 1094732 · Zbl 0729.58029

[47] Wojtkowski, M.