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Categorical Plücker formula and homological projective duality. (English) Zbl 1468.14038
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A. Kuznetsov in [Publ. Math., Inst. Hautes Étud. Sci. 105, 157–220 (2007; Zbl 1131.14017)] introduced Homological Projective Duality theory. For each smooth projective variety $X \subset \mathbb{P}^n$, he defines a semiorthogonal decomposition called Lefschetz decomposition, and the HP-dual X^\natural of X .

In the paper under review, the authors prove that if (X, X^\natural) and (T, T^\natural) are two HP-dual pairs, where $X, T \subset \mathbb{P}^n$ are smooth varieties intersecting properly, then the derived categories of coherent sheaves $D(X \times_{\mathbb{P}^n} T)$ and $D(X^\natural \times_{\mathbb{P}^n} T^\natural)$ admit semiorthogonal decompositions with isomorphic primitive parts. The primitive part is the component not coming from the ambient space. Kuznetsov's paper considered only the case where T is a linear subspace.

The first part of the article is a nice introduction to decomposition of derived categories and to HPD theory.

After the statement of the main result, the authors show some applications: counterexamples to the birational Torelli conjecture for Calabi-Yau threefolds and fivefolds, and a relation between the Euler characteristics of X, T , and their HP-duals.

The proof of the main result is long and technical, and expands Kuznetsov's original approach, which is well explained in [R. P. Thomas, Proc. Symp. Pure Math. 97, 585–609 (2018; Zbl 1451.14052)].

Let $\mathcal{H} \subset X \times T^\natural$ be defined by the incidence relation. The authors consider the embeddings $I_1 : D(X \times_{\mathbb{P}^n} T) \hookrightarrow D(\mathcal{H})$ and $I_2 : D(X^\natural \times_{\mathbb{P}^n} T^\natural) \hookrightarrow D(\mathcal{H})$. They show that the pull-back $I_1^* : D(\mathcal{H}) \rightarrow D(X \times_{\mathbb{P}^n} T)$ is fully faithful on some subcategories coming from the Lefschetz decompositions of $D(X)$ and $D(T)$. The same result holds for $I_2^* : D(\mathcal{H}) \rightarrow D(X^\natural \times_{\mathbb{P}^n} T^\natural)$, with the appropriate change of notation. The Lefschetz decompositions are not assumed to be rectangular. Finally, they prove that $I_2^* I_1$ gives the isomorphism between the principal parts of $D(X \times_{\mathbb{P}^n} T)$ and $D(X^\natural \times_{\mathbb{P}^n} T^\natural)$.

The underlying idea of the proof is to consider a decomposition of the subcategory $D(X \times_{\mathbb{P}^n} T)^\perp \subset D(\mathcal{H})$, where the pieces are nicely ordered in a “chess board”, and study the relations between each piece and the others pieces of the board.

Reviewer: [Giosuè Muratore \(Roma\)](#)

MSC:

- 14F08** Derived categories of sheaves, dg categories, and related constructions in algebraic geometry Cited in **6** Documents
- 18G80** Derived categories, triangulated categories

Keywords:

[categorification](#); [Plücker formula](#); [homological projective duality](#)

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