Braunack-Mayer, Vincent
Strict algebraic models for rational parametrised spectra. I. (English) Zbl 1468.55011

This paper develops a theory of algebraic models for the rational homotopy theory of parametrized spectra using Quillen’s classical point of view for rational homotopy theory.

D. Quillen [Ann. Math. (2), 90, 205–295 (1969; Zbl 0191.53702)] introduced the functors

\[ \lambda : \text{Top}_1 \to \text{DGL}_{\geq 1} \quad \text{and} \quad C : \text{Top}_1 \to \text{CDGC}_{\geq 2} \]

from the category \( \text{Top}_1 \) of simply-connected spaces to the categories \( \text{DGL}_{\geq 1} \) of positively graded dg Lie algebras, and \( \text{CDGC}_{\geq 2} \) of 2-connected cocommutative dg coalgebras, respectively.

Let \( X \) be a simply-connected space. Denote by

- \( \text{Mod}(\lambda(X)) \) the category of dg Lie modules over \( \lambda(X) \),
- \( \text{Mod}(C(X)) \) the category of dg comodules over \( C(X) \), and
- \( \text{Spec}_X \) the stable \( \infty \)-category of \( X \)-parametrized spectra.

Each of the categories above carries a certain model category structure (remark that in this work, the weak equivalences in \( \text{Mod}(C(X)) \) are those maps that become a quasi-isomorphism after applying the cobar construction, i.e., the \( \Omega \)-quasi-isomorphisms). The main result of this paper asserts that there are equivalences of homotopy categories

\[ \text{Ho}(\text{Spec}_X) \cong \text{Ho}(\text{Mod}(\lambda(X))) \cong \text{Ho}(\text{Mod}(C(X))) \].

Furthermore, the equivalences above are pseudonatural strongly symmetric monoidal with respect to the fiberwise smash product, the derived tensor product, and the derived cotensor product, respectively.

As an application of the main result, the author classifies the rational homotopy classes of fiberwise stable maps in terms of Ext and coExt groups as follows. For \( X \)-parametrized spectra \( P \) and \( Q \), denote by \( \{P, Q\}_X \) the \( \mathbb{Z} \)-graded abelian group of homotopy classes of fiberwise stable maps \( P \to Q \). If \( X \) is simply-connected, the author proves that:

1. If \( M, N \in \text{Mod}(\lambda(X)) \) correspond to the \( X \)-parametrized spectra \( P \) and \( Q \), respectively, then under the main equivalence of categories proven in the paper, there is an isomorphism of rational graded vector spaces
   \[ \{P, Q\}_X \otimes \mathbb{Q} \cong \text{Ext}^*_{U\lambda(X)}(M, N). \]
   Here, \( U \) is the universal enveloping algebra functor.

2. If \( M, N \in \text{Mod}(C(X)) \) correspond to the \( X \)-parametrized spectra \( P \) and \( Q \), respectively, then under the main equivalence of categories proven in the paper, there is an isomorphism of rational graded vector spaces
   \[ \{P, Q\}_X \otimes \mathbb{Q} \cong \text{coExt}^*_{C(X)}(M, N). \]

The paper is very well-written and detailed.

Similar results involving the category \( \text{Mod}(C^*(X; \mathbb{Q})) \) of dg modules over the associative dg algebra of cochains, with connections to Sullivan’s models, can be found in [Y. Félix et al., J. Topol., 3, 743–758 (2010; Zbl 1221.55012)].

In relation to Sullivan’s perspective of rational homotopy theory (using commutative dg algebras), this has been studied in a preprint by the author, [V. Braunack-Mayer, “Strict algebraic models for rational parametrised spectra II”, Preprint, arXiv:2011.06307]. There, under the usual finite \( \mathbb{Q} \)-type and nilpotency assumptions, the category of dg modules \( \text{Mod}(A) \) over a Sullivan cdga model \( A = (AV, d) \) of a space \( X \), there is a corresponding equivalence of homotopy categories with a sufficiently restricted category of \( X \)-parametrized spectra.

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55P62 Rational homotopy theory
16T15 Coalgebras and comodules; corings
55N99 Homology and cohomology theories in algebraic topology
55Q15 Whitehead products and generalizations

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