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A heuristic for boundedness of ranks of elliptic curves. (English) Zbl 1469.11173
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Summary: We present a heuristic that suggests that ranks of elliptic curves E over \mathbb{Q} are bounded. In fact, it suggests that there are only finitely many E of rank greater than 21. Our heuristic is based on modeling the ranks and Shafarevich-Tate groups of elliptic curves simultaneously, and relies on a theorem counting alternating integer matrices of specified rank. We also discuss analogues for elliptic curves over other global fields.

MSC:

11G05 Elliptic curves over global fields

11G40 L -functions of varieties over global fields; Birch-Swinnerton-Dyer conjecture

11P21 Lattice points in specified regions

14G25 Global ground fields in algebraic geometry

Cited in **2** Reviews
Cited in **16** Documents

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Software:

Magma; ecdata

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References:

- [1] Batyrev, V. V., Manin, Yu. I.: Sur le nombre des points rationnels de hauteur borné des variétés algébriques. *Math. Ann.* 286, 27-43 (1990)Zbl 0679.14008 MR 1032922 · Zbl 0679.14008
- [2] Bektimirov, B., Mazur, B., Stein, W., Watkins, M.: Average ranks of elliptic curves: tension between data and conjecture. *Bull. Amer. Math. Soc. (N.S.)* 44, 233-254 (2007)Zbl 1190.11032 MR 2291676 · Zbl 1190.11032
- [3] Bhargava, M., Kane, D. M., Lenstra, H. W., Jr., Poonen, B., Rains, E.: Modeling the distribution of ranks, Selmer groups, and Shafarevich-Tate groups of elliptic curves. *Cambridge J. Math.* 3, 275-321 (2015)Zbl 1329.14071 MR 3393023 · Zbl 1329.14071
- [4] Bhargava, M., Skinner, C., Zhang, W.: A majority of elliptic curves over \mathbb{Q} satisfy the Birch and Swinnerton-Dyer conjecture. *arXiv:1407.1826v2*(2014)
- [5] Brumer, A.: The average rank of elliptic curves. I. *Invent. Math.* 109, 445-472 (1992) Zbl 0783.14019 MR 1176198 · Zbl 0783.14019
- [6] Brumer, A., McGuinness, O.: The behavior of the Mordell-Weil group of elliptic curves. *Bull. Amer. Math. Soc. (N.S.)* 23, 375-382 (1990)Zbl 0741.14010 MR 1044170 · Zbl 0741.14010
- [7] Cassels, J. W. S.: Arithmetic on curves of genus 1. IV. Proof of the Hauptvermutung. *J. Reine Angew. Math.* 211, 95-112 (1962)Zbl 106.03706 MR 0163915 · Zbl 0106.03706
- [8] Cassels, J. W. S.: Diophantine equations with special reference to elliptic curves. *J. London Math. Soc.* 41, 193-291 (1966)Zbl 0138.27002 MR 0199150 · Zbl 0138.27002
- [9] Cohen, H., Lenstra, H. W., Jr.: Heuristics on class groups of number fields. In: *Number Theory, Noordwijkerhout 1983, Lecture Notes in Math.* 1068, Springer, Berlin, 33-62 (1984)Zbl 0558.12002 MR 756082
- [10] Cohen, H., Martinet, J.: Étude heuristique des groupes de classes des corps de nombres. *J. Reine Angew. Math.* 404, 39-76 (1990)Zbl 0699.12016 MR 1037430 · Zbl 0699.12016
- [11] Conrey, J. B., Ghosh, A.: Remarks on the generalized Lindelöf hypothesis. *Funct. Approx. Comment. Math.* 36, 71-78 (2006)Zbl 1196.11121 MR 2296639 · Zbl 1196.11121
- [12] Conrey, J. B., Keating, J. P., Rubinstein, M. O., Snaith, N. C.: On the frequency of vanishing of quadratic twists of modular L -functions. In: *Number Theory for the Millennium, I (Urbana, IL, 2000)*, A K Peters, Natick, MA, 301-315 (2002) Zbl 1044.11035 MR 1956231 · Zbl 1044.11035
- [13] Conrey, J. B., Rubinstein, M. O., Snaith, N. C., Watkins, M.: Discretisation for odd quadratic twists. In: *Ranks of Elliptic Curves and Random Matrix Theory, London Math. Soc. Lecture Note Ser.* 341, Cambridge Univ. Press, Cambridge, 201-214 (2007)Zbl 1193.11064 MR 2322346 A heuristic for boundedness of ranks of elliptic curves2899 · Zbl 1193.11064

- [14] Cornut, C.: Mazur's conjecture on higher Heegner points. *Invent. Math.* 148, 495-523 (2002)Zbl 111.11029 MR 1908058 · [Zbl 111.11029](#)
- [15] Cremona, J. E.: *Algorithms for Modular Elliptic Curves*. 2nd ed., Cambridge Univ. Press, Cambridge (1997)Zbl 0872.14041 MR 1628193 · [Zbl 0872.14041](#)
- [16] Dąbrowski, A., Jędrzejak, T., Szymaszkievicz, L.: Behaviour of the order of Tate-Shafarevich groups for the quadratic twists of $(X_0)(49)$. In: *Elliptic Curves, Modular Forms and Iwasawa Theory*, Springer Proc. Math. Statist. 188, Springer, Cham, 125- 159 (2016)Zbl 1398.11087 MR 3629650
- [17] Delaunay, C.: Heuristics on Tate-Shafarevich groups of elliptic curves defined over \mathbb{Q} . *Experiment. Math.* 10, 191-196 (2001)Zbl 045.11038 MR 1837670 · [Zbl 1045.11038](#)
- [18] Delaunay, C.: Moments of the orders of Tate-Shafarevich groups. *Int. J. Number Theory* 1, 243-264 (2005)Zbl 1082.11042 MR 2173383 · [Zbl 1082.11042](#)
- [19] Delaunay, C.: Heuristics on class groups and on Tate-Shafarevich groups: the magic of the Cohen-Lenstra heuristics. In: *Ranks of Elliptic Curves and Random Matrix Theory*, London Math. Soc. Lecture Note Ser. 341, Cambridge Univ. Press, Cambridge, 323-340 (2007)Zbl 1231.11129 MR 2322355 · [Zbl 1231.11129](#)
- [20] Delaunay, C., Duquesne, S.: Numerical investigations related to the derivatives of the L-series of certain elliptic curves. *Experiment. Math.* 12, 311-317 (2003) Zbl 1083.11041 MR 2034395 · [Zbl 1083.11041](#)
- [21] Delaunay, C., Jouhet, F.: p -torsion points in finite abelian groups and combinatorial identities. *Adv. Math.* 258, 13-45 (2014)Zbl 1286.11087 MR 3190422 · [Zbl 1286.11087](#)
- [22] Delaunay, C., Roblot, X.-F.: Regulators of rank one quadratic twists. *J. Théor. Nombres Bordeaux* 20, 601-624 (2008)Zbl 1201.11057 MR 2523310 · [Zbl 1201.11057](#)
- [23] Delaunay, C., Watkins, M.: The powers of logarithm for quadratic twists. In: *Ranks of Elliptic Curves and Random Matrix Theory*, London Math. Soc. Lecture Note Ser. 341, Cambridge Univ. Press, Cambridge, 189-193 (2007)Zbl 1213.11117 MR 2322344 · [Zbl 1213.11117](#)
- [24] Deninger, C.: The Hilbert-Pólya strategy and height pairings. In: *Casimir Force, Casimir Operators and the Riemann Hypothesis*, de Gruyter, Berlin, 275-283 (2010) Zbl 225.14018 MR 2777722
- [25] Dujella, A.: Infinite families of elliptic curves with high rank and prescribed torsion. <http://web.math.pmf.unizg.hr/~duje/tors/generic.html>(2017)
- [26] Duke, W., Rudnick, Z., Sarnak, P.: Density of integer points on affine homogeneous varieties. *Duke Math. J.* 71, 143-179 (1993)Zbl 0798.11024 MR 1230289 · [Zbl 0798.11024](#)
- [27] Durrett, R.: *Probability: Theory and Examples*. 4th ed. Cambridge Ser. Statist. Probab. Math., Cambridge Univ. Press, Cambridge (2010)Zbl 1202.60001 MR 2722836 · [Zbl 1202.60001](#)
- [28] Elkies, N. D.: Curves $Dy^2 = x^3 - x$ of odd analytic rank. In: *Algorithmic Number Theory (Sydney, 2002)*, Lecture Notes in Computer Sci. 2369, Springer, Berlin, 244- 251 (2002)Zbl 1058.11034 MR 2041088
- [29] Elkies, N. D.: $Z(2\sin E(\mathbb{Q}))$, etc. Email to nmbrrthry@listserv.nodak.edu (2006)
- [30] Eskin, A., Katznelson, Y. R.: Singular symmetric matrices. *Duke Math. J.* 79, 515- 547 (1995)Zbl 0832.11036 MR 1344769 · [Zbl 0832.11036](#)
- [31] Farmer, D. W., Gonek, S. M., Hughes, C. P.: The maximum size of L-functions. *J. Reine Angew. Math.* 609, 215-236 (2007)Zbl 1234.11109 MR 2350784 · [Zbl 1234.11109](#)
- [32] Fouvry, É., Klüners, J.: On the 4-rank of class groups of quadratic number fields. *Invent. Math.* 167, 455-513 (2007)Zbl 1126.11062 MR 2276261
- [33] Friedman, E., Washington, L. C.: On the distribution of divisor class groups of curves over a finite field. In: *Théorie des nombres (Quebec, PQ, 1987)*, de Gruyter, Berlin, 227-239 (1989)Zbl 0693.12013 MR 1024565 Jennifer Park et al.
- [34] Furstenberg, H., Katznelson, Y.: A density version of the Hales-Jewett theorem. *J. Anal. Math.* 57, 64-119 (1991)Zbl 0770.05097 MR 1191743 · [Zbl 0770.05097](#)
- [35] Gerth, F., III: Extension of conjectures of Cohen and Lenstra. *Exposition. Math.* 5, 181-184 (1987)Zbl 0613.12003 MR 887792 · [Zbl 0613.12003](#)
- [36] Goldfeld, D.: Conjectures on elliptic curves over quadratic fields. In: *Number Theory, Carbondale 1979*, Lecture Notes in Math. 751, Springer, Berlin, 108-118 (1979) Zbl 0417.14031 MR 564926
- [37] Goldfeld, D., Szpiro, L.: Bounds for the order of the Tate-Shafarevich group. *Compos. Math.* 97, 71-87 (1995)Zbl 860.11032 MR 1355118 · [Zbl 0860.11032](#)
- [38] Granville, A.: Harald Cramér and the distribution of prime numbers. *Scand. Actuar. J.* 1995, 12-28Zbl 0833.01018 MR 1349149 · [Zbl 0833.01018](#)
- [39] Granville, A.: ABC allows us to count squarefrees. *Int. Math. Res. Notices* 1998, 991-1009Zbl 0924.11018 MR 1654759 · [Zbl 0924.11018](#)
- [40] Gross, B. H.: Heights and the special values of L-series. In: *Number Theory (Montreal, Que., 1985)*, CMS Conf. Proc. 7, Amer. Math. Soc., Providence, RI, 115- 187 (1987)Zbl 0623.10019 MR 894322
- [41] Hall, P.: A partition formula connected with Abelian groups. *Comment. Math. Helv.* 11, 126-129 (1938)Zbl 0019.39705 MR 1509594 · [Zbl 64.0060.04](#)
- [42] Harron, R., Snowden, A.: Counting elliptic curves with prescribed torsion. *J. Reine Angew. Math.* 729, 151-170 (2017)Zbl 1378.11067 MR 3680373 · [Zbl 1378.11067](#)
- [43] Hindry, M.: Why is it difficult to compute the Mordell-Weil group? In: *Diophantine Geometry*, CRM Ser. 4, Ed. Normale,

- Pisa, 197-219 (2007)Zbl 1219.11099 MR 2349656 · [Zbl 1219.11099](#)
- [44] Hindry, M., Pacheco, A.: An analogue of the Brauer-Siegel theorem for abelian varieties in positive characteristic. *Moscow Math. J.* 16, 45-93 (2016)Zbl 1382.11041 MR 3470576 · [Zbl 1382.11041](#)
- [45] Hindry, M., Silverman, J. H.: The canonical height and integral points on elliptic curves. *Invent. Math.* 93, 419-450 (1988)Zbl 0657.14018 MR 948108 · [Zbl 0657.14018](#)
- [46] Honda, T.: Isogenies, rational points and section points of group varieties. *Japan. J. Math.* 30, 84-101 (1960)Zbl 0109.39602 MR 0155828 · [Zbl 0109.39602](#)
- [47] Hortsch, R.: Counting elliptic curves of bounded Faltings height.arXiv:1505.05112v1 (2015) · [Zbl 1355.11068](#)
- [48] Iwaniec, H., Sarnak, P.: Perspectives on the analytic theory of L-functions. In: *GAF A 2000 (Tel Aviv, 1999)*, *Geom. Funct. Anal.*, Special Volume, 705-741 (2000) Zbl 0996.11036 MR 1826269 · [Zbl 0996.11036](#)
- [49] Katz, N. M., Sarnak, P.: *Random Matrices, Frobenius Eigenvalues, and Monodromy*. *Amer. Math. Soc. Colloq. Publ.* 45, Amer. Math. Soc., Providence, RI (1999) Zbl 0958.11004 MR 1659828 · [Zbl 0958.11004](#)
- [50] Katz, N. M., Sarnak, P.: Zeroes of zeta functions and symmetry. *Bull. Amer. Math. Soc. (N.S.)* 36, 1-26 (1999)Zbl 0921.11047 MR 1640151 · [Zbl 0921.11047](#)
- [51] Keating, J. P., Snaith, N. C.: Random matrix theory and L-functions at $s = 1/2$. *Comm. Math. Phys.* 214, 91-110 (2000)Zbl 1051.11047 MR 1794267 · [Zbl 1051.11047](#)
- [52] Kohnen, W., Zagier, D.: Values of L-series of modular forms at the center of the critical strip. *Invent. Math.* 64, 175-198 (1981)Zbl 0468.10015 MR 629468 · [Zbl 0468.10015](#)
- [53] Kurchanov, P. F.: The zeta-function of elliptic curves over certain abelian extensions of imaginary quadratic fields. *Mat. Sb. (N.S.)* 102 (144), 56-70, 151 (1977) (in Russian)Zbl 0443.12004MR 0498391 · [Zbl 0443.12004](#)
- [54] Lang, S.: Conjectured Diophantine estimates on elliptic curves. In: *Arithmetic and Geometry, Vol. I*, *Progr. Math.* 35, Birkhäuser Boston, Boston, MA, 155-171 (1983) Zbl 0529.14017 MR 717593 A heuristic for boundedness of ranks of elliptic curves2901 · [Zbl 0529.14017](#)
- [55] Lang, S.: Old and new conjectured Diophantine inequalities. *Bull. Amer. Math. Soc. (N.S.)* 23, 37-75 (1990)Zbl 0714.11034 MR 1005184 · [Zbl 0714.11034](#)
- [56] Lang, S.: *Algebraic Number Theory*. 2nd ed. *Grad. Texts in Math.* 110, Springer, New York (1994)Zbl 0811.11001 MR 1282723 · [Zbl 0811.11001](#)
- [57] Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. The user language. *J. Symbolic Comput.* 24, 235-265 (1997)Zbl 0898.68039 MR 1484478 · [Zbl 0898.68039](#)
- [58] Mazur, B.: Modular curves and the Eisenstein ideal. *Inst. Hautes Études Sci. Publ. Math.* 47, 33-186 (1978) (1977)Zbl 0394.14008 MR 488287 · [Zbl 0394.14008](#)
- [59] Mazur, B.: Modular curves and arithmetic. In: *Proc. International Congress of Mathematicians (Warszawa, 1983)*, Vol. 1, PWN, Warszawa, 185-211 (1984) Zbl 0597.14023 MR 804682
- [60] Merel, L.: Bornes pour la torsion des courbes elliptiques sur les corps de nombres. *Invent. Math.* 124, 437-449 (1996)Zbl 0936.11037 MR 1369424 · [Zbl 0936.11037](#)
- [61] Mestre, J.-F.: Construction d'une courbe elliptique de rang ≥ 12 . *C. R. Acad. Sci. Paris Sér. I Math.* 295, 643-644 (1982)Zbl 0541.14027 MR 688896 · [Zbl 0541.14027](#)
- [62] Mestre, J.-F.: Formules explicites et minoration de conducteurs de variétés algébriques. *Compos. Math.* 58, 209-232 (1986)Zbl 0607.14012 MR 844410 · [Zbl 0607.14012](#)
- [63] Mordell, L. J.: On the rational solutions of the indeterminate equations of the third and fourth degrees. *Proc. Cambridge Philos. Soc.* 21, 179-192 (1922) · [Zbl 48.0140.03](#)
- [64] Néron, A.: Problèmes arithmétiques et géométriques rattachés à la notion de rang d'une courbe algébrique dans un corps. *Bull. Soc. Math. France* 80, 101-166 (1952) Zbl 0049.30803 MR 0056951 · [Zbl 0049.30803](#)
- [65] Néron, A.: Propriétés arithmétiques de certaines familles de courbes algébriques. In: *Proc. International Congress of Mathematicians (Amsterdam, 1954)*, Vol. III, Noordhoff, Groningen, and North-Holland, Amsterdam, 481-488 (1956)Zbl 0074.15901 MR 0087210
- [66] Ono, K., Skinner, C.: Non-vanishing of quadratic twists of modular L-functions. *Invent. Math.* 134, 651-660 (1998)Zbl 0937.11017 MR 1660945 · [Zbl 0937.11017](#)
- [67] Poincaré, H.: Sur les propriétés arithmétiques des courbes algébriques. *J. Math. Pures Appl. (5)* 7, 161-234 (1901)JFM 32.0564.06 · [Zbl 32.0564.06](#)
- [68] Poincaré, H.: *Œuvres d'Henri Poincaré, Volume 5*. Gauthier-Villars, Paris (1950) Zbl 0041.37403
- [69] Polymath, D. H. J.: A new proof of the density Hales-Jewett theorem. *Ann. of Math. (2)* 175, 1283-1327 (2012)Zbl 1267.11010 MR 2912706 · [Zbl 1267.11010](#)
- [70] Poonen, B.: Squarefree values of multivariable polynomials. *Duke Math. J.* 118, 353-373 (2003)Zbl 1047.11021 MR 1980998 · [Zbl 1047.11021](#)
- [71] Poonen, B., Rains, E.: Random maximal isotropic subspaces and Selmer groups. *J. Amer. Math. Soc.* 25, 245-269 (2012)Zbl 1294.11097 MR 2833483 · [Zbl 1294.11097](#)
- [72] Poonen, B., Stoll, M.: The Cassels-Tate pairing on polarized abelian varieties. *Ann. of Math. (2)* 150, 1109-1149 (1999)Zbl 1024.11040 MR 1740984 · [Zbl 1024.11040](#)
- [73] Rubin, K., Silverberg, A.: Ranks of elliptic curves in families of quadratic twists. *Experiment. Math.* 9, 583-590 (2000)Zbl

- 0959.11023 MR 1806293 · [Zbl 0959.11023](#)
- [74] Rubin, K., Silverberg, A.: Rank frequencies for quadratic twists of elliptic curves. *Experiment. Math.* 10, 559-569 (2001)Zbl 1035.11025 MR 1881757 · [Zbl 1035.11025](#)
- [75] Rubin, K., Silverberg, A.: Ranks of elliptic curves. *Bull. Amer. Math. Soc. (N.S.)* 39, 455-474 (2002)Zbl 1052.11039 MR 1920278 · [Zbl 1052.11039](#)
- [76] Sarnak, P.: *Some Applications of Modular Forms*. Cambridge Tracts in Math. 99, Cambridge Univ. Press, Cambridge (1990)Zbl 1141.11023 MR 1102679 2902Jennifer Park et al.
- [77] Schmidt, W. M.: Asymptotic formulae for point lattices of bounded determinant and subspaces of bounded height. *Duke Math. J.* 35, 327-339 (1968)Zbl 0172.06304 MR 0224562 · [Zbl 0172.06304](#)
- [78] Serre, J.-P.: *Lectures on the Mordell-Weil Theorem*. 3rd ed., Aspects of Math., Vieweg, Braunschweig (1997)Zbl 0863.14013 MR 1757192
- [79] Shioda, T.: Some remarks on elliptic curves over function fields. In: *Journées Arithmétiques (Genève, 1991)*, Astérisque 209, 12, 99-114 (1992)Zbl 0820.14016 MR 1211006 · [Zbl 0820.14016](#)
- [80] Silverman, J. H.: Lower bound for the canonical height on elliptic curves. *Duke Math. J.* 48, 633-648 (1981)Zbl 0475.14033 MR 630588 · [Zbl 0475.14033](#)
- [81] Silverman, J. H.: Heights and the specialization map for families of abelian varieties. *J. Reine Angew. Math.* 342, 197-211 (1983)Zbl 0505.14035 MR 703488 · [Zbl 0505.14035](#)
- [82] Silverman, J. H.: Heights and elliptic curves. In: *Arithmetic Geometry (Storrs, CN, 1984)*, Springer, New York, 253-265 (1986)Zbl 0603.14020 MR 861979
- [83] Silverman, J. H.: *The Arithmetic of Elliptic Curves*. Grad. Texts in Math. 106, Springer, New York (1992)MR 1329092
- [84] Silverman, J. H.: *The Arithmetic of Elliptic Curves*. 2nd ed. Grad. Texts in Math. 106, Springer, Dordrecht (2009)Zbl 1194.11005 MR 2514094 · [Zbl 1194.11005](#)
- [85] Stein, W., Wuthrich, C.: Algorithms for the arithmetic of elliptic curves using Iwasawa theory. *Math. Comp.* 82, 1757-1792 (2013)Zbl 1336.11047 MR 3042584 · [Zbl 1336.11047](#)
- [86] Stewart, C. L., Top, J.: On ranks of twists of elliptic curves and power-free values of binary forms. *J. Amer. Math. Soc.* 8, 943-973 (1995)Zbl 0857.11026 MR 1290234 · [Zbl 0857.11026](#)
- [87] Tate, J. T.: The arithmetic of elliptic curves. *Invent. Math.* 23, 179-206 (1974) Zbl 0296.14018 MR 0419359 · [Zbl 0296.14018](#)
- [88] Tate, J.: On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. In: *Séminaire Bourbaki, Vol. 9, exp. 306*, Soc. Math. France, Paris, 415-440 (1995) Zbl 0199.55604 MR 1610977
- [89] Tate, J., Shafarevich, I. R.: The rank of elliptic curves. *Dokl. Akad. Nauk SSSR* 175, 770-773 (1967) (in Russian)Zbl 0168.42201 MR 0237508 · [Zbl 0168.42201](#)
- [90] Ulmer, D.: Elliptic curves with large rank over function fields. *Ann. of Math. (2)* 155, 295-315 (2002)Zbl 1109.11314 MR 1888802 · [Zbl 1109.11314](#)
- [91] Vatsal, V.: Special values of anticyclotomic L-functions. *Duke Math. J.* 116, 219-261 (2003)Zbl 1065.11048 MR 1953292 · [Zbl 1065.11048](#)
- [92] Venkatesh, A., Ellenberg, J. S.: Statistics of number fields and function fields. In: *Proc. International Congress of Mathematicians (Allahabad, 2010)*, Vol. II, Hindustan Book Agency, New Delhi, 383-402 (2010)Zbl 1259.11106 MR 2827801 · [Zbl 1259.11106](#)
- [93] Waldspurger, J.-L.: Sur les coefficients de Fourier des formes modulaires de poids demi-entier. *J. Math. Pures Appl. (9)* 60, 375-484 (1981)Zbl 0431.10015 MR 646366 · [Zbl 0431.10015](#)
- [94] Wang, Y., Stanley, R. P.: The Smith normal form distribution of a random integer matrix. *SIAM J. Discrete Math.* 31, 2247-2268 (2017)Zbl 1372.05006 MR 3706911 · [Zbl 1372.05006](#)
- [95] Watkins, M.: Some heuristics about elliptic curves. *Experiment. Math.* 17, 105-125 (2008)Zbl 1151.14025 MR 2410120 · [Zbl 1151.14025](#)
- [96] Watkins, M.: On elliptic curves and random matrix theory. *J. Théor. Nombres Bordeaux* 20, 829-845 (2008)Zbl 1193.11058 MR 2523320 · [Zbl 1193.11058](#)
- [97] Watkins, M.: A discursus on 21 as a bound for ranks of elliptic curves over \mathbb{Q} , and sundry related topics.<http://magma.maths.usyd.edu.au/~watkins/DISCURSUS.pdf>(2015) A heuristic for boundedness of ranks of elliptic curves2903 [Wat+14]Watkins, M., Donnelly, S., Elkies, N. D., Fisher, T., Granville, A., Rogers, N. F.: Ranks of quadratic twists of elliptic curves. *Publ. Math. Besançon* 2014/2, 63-98 (2014)Zbl 1365.11070 MR 3381037 · [Zbl 1365.11070](#)
- [98] de Weger, B. M. M.: $A + B = C$ and big X 's. *Quart. J. Math. Oxford Ser. (2)* 49, 105-128 (1998)Zbl 0917.11020 MR 1617347
- [99] Wiles, A.: The Birch and Swinnerton-Dyer conjecture. In: *The Millennium Prize Problems*, Clay Math. Inst., Cambridge, MA, 31-41 (2006)Zbl 1194.11006 MR 2238272 · [Zbl 1194.11006](#)
- [100] Wong, S.: On the density of elliptic curves. *Compos. Math.* 127, 23-54 (2001) Zbl 1003.11023 MR 1832985 · [Zbl 1003.11023](#)
- [101] Wood, M. M.: The distribution of sandpile groups of random graphs. *J. Amer. Math. Soc.* 30, 915-958 (2017)Zbl 1366.05098 MR 3671933 · [Zbl 1366.05098](#)
- [102] Wood, M. M.: Random integral matrices and the Cohen-Lenstra heuristics. *Amer. J. Math.* 141, 383-398 (2019)Zbl 07057016 MR 3928040 · [Zbl 1446.11170](#)
- [103] Young, M. P.: On the non-vanishing of elliptic curve L-functions at the central point. *Proc. London Math. Soc. (3)* 93, 1-42 (2006)Zbl 1174.11052 MR 2235480 · [Zbl 1174.11052](#)

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