

Blum, Harold; Liu, Yuchen

The normalized volume of a singularity is lower semicontinuous. (English) Zbl 1470.14008
J. Eur. Math. Soc. (JEMS) 23, No. 4, 1225-1256 (2021).

Let X be an n -dimensional normal variety and let D be an effective divisor on X . The pair (X, D) is klt if $(K_X + D)$ is \mathbb{Q} -Cartier and, for some log resolution $f : Y \rightarrow (X, D)$, the divisor $K_Y - f^*(K_X + D)$ has all coefficients > -1 . Given a closed point $x \in X$, we call the data (x, X, D) a klt singularity. Inspired by the study of Sasaki-Einstein metrics, and motivated by the study of Kähler-Einstein metrics on Fano varieties, Chi Li introduced in [*C. Li*, *Math. Z.* 289, No. 1–2, 491–513 (2018; [Zbl 1423.14025](#))] the notion of normalized volume $\widehat{\text{vol}}(x, X, D)$ of a klt singularity (x, X, D) . In the paper under review, the authors show that given a \mathbb{Q} -Gorenstein flat family of klt singularities (x_t, X_t, D_t) over a normal variety T , the normalized volume $\widehat{\text{vol}}(x_t, X_t, D_t)$ is lower semicontinuous in t , with respect to the Zariski topology.

Such a result has several notable applications, either in the study of klt singularity themselves, or in the study of \mathbb{Q} -Fano varieties, via the affine cone construction (the affine cone over a Fano variety defined by a multiple of the anticanonical Kodaira embedding provides an archetype of klt singularity). First, a direct application shows that smooth points have the largest possible normalized volume, equal to n^n , among all klt singularities. This statement was also proved in [Theorem A.4, *Y. Liu* and *Ch Xu*, *Duke Math. J.* 168, No. 11, 2029–2073 (2019; [Zbl 1436.14085](#))], but the paper under review proposes some generalizations along this line. Second, and maybe most importantly, the authors prove Zariski openness of log K-semistability under a conjectural hypothesis of constructibility of $t \mapsto \widehat{\text{vol}}(x_t, X_t, D_t)$. The latter constructibility property was subsequently proved by *Ch. Xu* [*Ann. Math. (2)* 191, No. 3, 1003–1030 (2020; [Zbl 1469.14033](#))]. Finally, applications to the study of Gromov-Hausdorff limits of Fano Kähler-Einstein manifolds are given, where lower semicontinuity of the normalized volume translates as lower semicontinuity of the volume density function.

The method of proof used by the authors is well described in the short introduction to the article, with a clear description of the structure of the paper.

Reviewer: [Thibaut Delcroix \(Montpellier\)](#)

MSC:

[14B05](#) Singularities in algebraic geometry
[13A18](#) Valuations and their generalizations for commutative rings
[32Q20](#) Kähler-Einstein manifolds

Cited in **1** Review
Cited in **4** Documents

Keywords:

[singularities](#); [normalized volume](#); [K-stability](#)

Full Text: [DOI](#) [arXiv](#)

References:

- [1] Ambro, F.: Variation of log canonical thresholds in linear systems. *Int. Math. Res. Notices*2016, 4418-4448Zbl 1404.14020 MR 3556423 · [Zbl 1404.14020](#)
- [2] Bhatt, B., Gabber, O., Olsson, M.: Finiteness of étale fundamental groups by reduction modulop.arXiv:1705.07303(2017)
- [3] Blum, H.: Existence of valuations with smallest normalized volume. *Compos. Math.* 154, 820-849 (2018)Zbl 1396.14007 MR 3778195 · [Zbl 1396.14007](#)
- [4] Blum, H., Liu, Y.: Openness of uniform K-stability in families of \mathbb{Q} -Fano varieties. *Ann. Sci. École Norm. Sup.*, to appear;arXiv:1808.09070(2018)
- [5] Boucksom, S., Favre, C., Jonsson, M.: A refinement of Izumi’s theorem. In: *Valuation Theory in Interaction*, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 55-81 (2014) Zbl 1312.13006 MR 3329027 · [Zbl 1312.13006](#)
- [6] Cutkosky, S.: Multiplicities associated to graded families of ideals. *Algebra Number Theory*7, 2059-2083 (2013)Zbl 1315.13040 MR 3152008 · [Zbl 1315.13040](#)
- [7] de Fernex, T., Ein, L., Mustat, Şa, M.: Multiplicities and log canonical threshold. *J. Algebraic Geom.*13, 603-615 (2004)Zbl

- 1068.14006 MR 2047683 · Zbl 1068.14006
- [8] Donaldson, S.: Algebraic families of constant scalar curvature Kähler metrics. In: Surveys in Differential Geometry 2014: Regularity and Evolution of Nonlinear Equations, *Surv. Differ. Geom.* 19, Int. Press, Somerville, MA, 111-137 (2015) Zbl 1330.53093 MR 3381498 · Zbl 1330.53093
- [9] Ein, L., Lazarsfeld, R., Smith, K.: Uniform approximation of Abhyankar valuation ideals in smooth function fields. *Amer. J. Math.* 125, 409-440 (2003) Zbl 1033.14030 MR 1963690 · Zbl 1033.14030
- [10] Flenner, H., Manaresi, M.: Equimultiplicity and equidimensionality of normal cones. In: Recent Progress in Intersection Theory (Bologna, 1997), *Trends Math.*, Birkhäuser Boston, Boston, MA, 199-215 (2000) Zbl 1081.13506 MR 1849295 · Zbl 1081.13506
- [11] Fujita, K.: Optimal bounds for the volumes of Kähler-Einstein Fano manifolds. *Amer. J. Math.* 140, 391-414 (2018) Zbl 1400.14105 MR 3783213 · Zbl 1400.14105
- [12] Hein, H., Sun, S.: Calabi-Yau manifolds with isolated conical singularities. *Publ. Math. Inst. Hautes Études Sci.* 126, 73-130 (2017) Zbl 1397.32009 MR 3735865 · Zbl 1397.32009
- [13] Jonsson, M., Mustaţǎ, M.: Valuations and asymptotic invariants for sequences of ideals. *Ann. Inst. Fourier (Grenoble)* 62, 2145-2209 (2012) Zbl 1272.14016 MR 3060755 · Zbl 1272.14016
- [14] Kaveh, K., Khovanskii, A.: Convex bodies and multiplicities of ideals. *Proc. Steklov Inst. Math.* 286, 268-284 (2014) Zbl 1315.13013 MR 3482603 · Zbl 1315.13013
- [15] Kollár, J.: Rational Curves on Algebraic Varieties. *Ergeb. Math. Grenzgeb.* 32, Springer, Berlin (1996) Zbl 0877.14012 MR 1440180
- [16] Kollár, J.: Seifert Gm-bundles. [arXiv:math/0404386](https://arxiv.org/abs/math/0404386)(2004)
- [17] Kollár, J.: Singularities of the Minimal Model Program. *Cambridge Tracts in Math.* 200, Cambridge Univ. Press, Cambridge (2013) Zbl 1282.14028 MR 3057950
- [18] Lazarsfeld, R.: Positivity in Algebraic Geometry. I-II. *Ergeb. Math. Grenzgeb.* 4849, Springer, Berlin (2004) Zbl 1093.14501(I) Zbl 1093.14500(II) MR 2095471(I) MR 2095472(II) · Zbl 1066.14021
- [19] Lazarsfeld, R., Mustaţǎ, M.: Convex bodies associated to linear series. *Ann. Sci. École Norm. Sup. (4)* 42, 783-835 (2009) Zbl 1182.14004 MR 2571958 · Zbl 1182.14004
- [20] Lech, C.: Note on multiplicities of ideals. *Ark. Mat.* 4, 63-86 (1960) Zbl 0192.13901 MR 0140536 · Zbl 0192.13901
- [21] Li, C.: K-semistability is equivariant volume minimization. *Duke Math. J.* 166, 3147- 3218 (2017) Zbl 1409.14008 MR 3715806 · Zbl 1409.14008
- [22] Li, C.: Yau-Tian-Donaldson correspondence for K-semistable Fano manifolds. *J. Reine Angew. Math.* 733, 55-85 (2017) Zbl 1388.53076 MR 3731324 · Zbl 1388.53076
- [23] Li, C.: Minimizing normalized volumes of valuations. *Math. Z.* 289, 491-513 (2018) Zbl 1423.14025 MR 3803800 · Zbl 1423.14025
- [24] Li, C., Liu, Y.: Kähler-Einstein metrics and volume minimization. *Adv. Math.* 341, 440-492 (2019) Zbl 1404.32044 MR 3872852 · Zbl 1404.32044
- [25] Li, C., Wang, X., Xu, C.: On the proper moduli space of smoothable Kähler-Einstein Fano varieties. *Duke Math. J.* 168, 1387-1459 (2019) Zbl 07080115 MR 3959862 · Zbl 1469.14087
- [26] Li, C., Xu, C.: Stability of valuations and Kollár components. *J. Eur. Math. Soc.* 22, 2573-2627 (2020) Zbl 07227742 MR 4118616 · Zbl 1471.14076
- [27] Li, C., Xu, C.: Stability of valuations: higher rational rank. *Peking Math. J.* 1, 1-79 (2018) Zbl 1423.14262 MR 4059992 · Zbl 1423.14262
- [28] Liu, Y.: The volume of singular Kähler-Einstein Fano varieties. *Compos. Math.* 154, 1131-1158 (2018) Zbl 1397.14052 MR 3797604 · Zbl 1397.14052
- [29] Liu, Y., Xu, C.: K-stability of cubic threefolds. *Duke Math. J.* 168, 2029-2073 (2019) Zbl 1436.14085 MR 3992032 · Zbl 1436.14085
- [30] Matsumura, H.: Commutative Algebra. 2nd ed., *Math. Lecture Note Ser.* 56, Benjamin/Cummings, Reading MA (1980) Zbl 0441.13001 MR 0575344 · Zbl 0441.13001
- [31] Mustaţǎ, M.: On multiplicities of graded sequences of ideals. *J. Algebra* 256, 229-249 (2002) Zbl 1076.13500 MR 1936888
- [32] Odaka, Y.: On the moduli of Kähler-Einstein Fano manifolds. In: Proceedings of Kinosaki Algebraic Geometry Symposium (2013)
- [33] Odaka, Y.: Compact moduli spaces of Kähler-Einstein Fano varieties. *Publ. RIMS Kyoto Univ.* 51, 549-565 (2015) Zbl 1333.14039 MR 3395458 · Zbl 1333.14039
- [34] Spotti, C., Sun, S.: Explicit Gromov-Hausdorff compactifications of moduli spaces of Kähler-Einstein Fano manifolds. *Pure Appl. Math. Quart.* 13, 477-515 (2017) Zbl 1403.32013 MR 3882206 · Zbl 1403.32013
- [35] Spotti, C., Sun, S., Yao, C.: Existence and deformations of Kähler-Einstein metrics on smoothable Q-Fano varieties. *Duke Math. J.* 165, 3043-3083 (2016) Zbl 1362.53082 MR 3566198 · Zbl 1362.53082
- [36] The Stacks Project Authors: Stacks Project. <http://stacks.math.columbia.edu>
- [37] Xu, C.: Finiteness of algebraic fundamental groups. *Compos. Math.* 150, 409-414 (2014) Zbl 1291.14057 MR 3187625 · Zbl 1291.14057
- [38] Xu, C.: Interaction between singularity theory and the minimal model program. In: Proc. Int. Congress Math. 2018, Rio de

This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.