Freixas i Montplet, Gerard; von Pippich, Anna-Maria

Riemann-Roch isometries in the non-compact orbifold setting. (English) Zbl 1470.14021

J. Eur. Math. Soc. (JEMS) 22, No. 11, 3491-3564 (2020).

This work is a generalization to Riemann surfaces $X_\Gamma = \Gamma\backslash\mathbb{H}^2$ of a Grothendieck-Riemann-Roch formula established by *P. Deligne* [Contemp. Math. 67, 93–117 (1987; Zbl 0629.14008)] and *J.-M. Bismut*, *H. Gillet* and *C. Soulé* [Commun. Math. Phys. 115, No. 1, 49–78, 79–126; No. 2, 301–351 (1988; Zbl 0651.32017)]. Here, Γ is a Fuchsian group of the first kind with singularities of cusp or orbifold type (corresponding to parabolic and elliptic elements of Γ). The main theorem relates the determinant of some cohomology space with a Quillen type metric, Selberg zeta function values and geometry of bundles on moduli spaces of pointed orbicurves. The novelty of this important work is to permit singularities on the surface X_Γ: the results of this paper apply to the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ in particular. The core of the paper consists of deep analytic surgery techniques through Mayer-Vietoris formulas for determinants of Laplacians introduced by *D. Burghelea*, *L. Friedlander* and *L. Kappeler* [J. Funct. Anal. 107, No. 1, 34–65 (1992; Zbl 0759.58043)]. With appropriate domains obtained by taking off small ε-neighborhood around every singularity and performing the spectral analysis for the $\varepsilon \to 0$ limits, the authors reduce the determinants for the Laplace-Beltrami or Dirichlet-to-Neumann operators to explicit expressions for model hyperbolic cusps and cones. Such computations, which appear notably in theoretical physics literature, are developed here with full mathematical rigor; they can be used for several variants in the geometry of bundles on moduli spaces, such as the Burgos-Kramer-Kühn program of extending arithmetic intersection theory to singular Hermitian vector bundles (see *U. Kühn* [J. Reine Angew. Math. 534, 209–236 (2001; Zbl 1084.14028)]). As consequences of the main theorem, let us quote the proof of an arithmetic Riemann-Roch formula in the realm of Arakelov geometry and an expression of the Selberg zeta special value $Z'(1, \mathrm{PSL}_2(\mathbb{Z}))$ in terms of logarithmic derivatives of Dirichlet L-function values.

Reviewer: Laurent Guillopé (Nantes)

MSC:

14C40 Riemann-Roch theorems
14G40 Arithmetic varieties and schemes; Arakelov theory; heights
11F72 Spectral theory; trace formulas (e.g., that of Selberg)
58J52 Determinants and determinant bundles, analytic torsion
14F43 Other algebro-geometric (co)homologies (e.g., intersection, equivariant, Lawson, Deligne (co)homologies)

Keywords:

Riemann-Roch theorem; Arakelov geometry; Fuchsian group; Quillen metric; analytic surgery; Mayer-Vietoris; Dirichlet-to-Neumann; Selberg zeta function; L-functions

Full Text: DOI arXiv

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