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Global Frobenius liftability. I. (English) Zbl 1471.14055
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The paper addresses the problem of characterizing F-liftable smooth projective varieties in positive characteristic. Here, a variety X over a perfect field k of positive characteristic is *F-liftable* if it admits a model $\widetilde{X}$ over the ring of truncated Witt vectors $W_2(k)$ together with a model $\widetilde{F}_X : \widetilde{X} \to \widetilde{X}$ of the Frobenius morphism F_X of X.

The paper puts forward the following conjecture.

> **Conjecture:** Every F-liftable smooth projective variety over an algebraically closed field of positive characteristic has a finite étale cover whose Albanese morphism is a toric fibration.

The first main result relates the conjecture to a conjecture by *G. Occhetta* and *J. A. Wiśniewski* [Math. Z. 241, No. 1, 35–44 (2002; Zbl 1022.14016)]:

> If every smooth projective simply connected F-liftable variety over an algebraically closed field of positive characteristic is a toric variety, then every smooth projective variety X in characteristic zero with a surjective morphism $Y \to X$ from a complete toric variety Y is a toric variety.

The proof is based on a descent result for F-liftability along morphisms of finite type, and an investigation of toric varieties in families:

- Given a smooth projective family $X \to S$ such that the fiber X_s is a toric variety for a dense set of $s \in S$, if the generic fiber has characteristic 0 then it is a toric variety.
- Given a relative normal crossing pair over a DVR, if the generic fiber is a toric pair, then so is the special fiber.
- A relative normal crossing pair is a toric fibration if and only if any fiber over a geometric point is a toric pair.

As a byproduct of the proof, every smooth projective variety X in characteristic zero with a surjective morphism $Y \to X$ from a complete toric variety Y satisfies Bott vanishing, as F-liftable varieties satisfy Bott vanishing by *A. Buch* et al. [Tohoku Math. J. (2) 49, No. 3, 355–366 (1997; Zbl 0899.14026)].

The second main result characterizes projective F-liftable normal crossing pairs (X, D) with numerically trivial log canonical bundle $\omega_X(D)$ by showing that they are exactly the varieties X that admit a finite étale cover whose Albanese map is a toric fibration over an abelian variety. This generalizes the result of *V. B. Mehta* and *V. Srinivas* [Compos. Math. 64, 191–212 (1987; Zbl 0639.14024)] for F-liftable varieties with numerically trivial canonical bundle to the logarithmic setting.

Finally, the paper shows that the Conjecture holds for homogeneous spaces, thus proving a conjectural characterization of F-liftable rational homogeneous spaces by *A. Buch* et al. [Tohoku Math. J. (2) 49, No. 3, 355–366 (1997; Zbl 0899.14026)]. More precisely, the paper shows that over an algebraically closed field of positive characteristic:

- The only F-liftable smooth Fano varieties with nef tangent bundle and Picard rank 1 are projective spaces.
- Every F-liftable homogeneous space is isomorphic to a product of projective spaces and an ordinary abelian variety.

The first result is deduced from *S. Mori*'s characterization of the projective space [Ann. Math. (2) 110, 593–606 (1979; Zbl 0423.14006)] via a careful analysis of the restriction of differential forms invariant under $\frac{d\widetilde{F}_X}{p} : F_X^*\Omega_X^1 \to \Omega_X^1$ to rational curves in a special covering family. The second result relies on the classification of homogeneous spaces and their characterization in terms of marked Dynkin diagrams.

The authors illustrate their results and highlight the nuances with plenty of examples. They also announce that F-liftable varieties of small dimension are investigated in the paper *Global Frobenius liftability II:*

surfaces and Fano threefolds by the same authors.

Reviewer: Marta Pieropan (Utrecht)

MSC:

14G17 Positive characteristic ground fields in algebraic geometry
14M17 Homogeneous spaces and generalizations
14M25 Toric varieties, Newton polyhedra, Okounkov bodies
14J45 Fano varieties

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Frobenius lifting; toric variety; abelian variety; trivial log tangent bundle

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References:

[1] Achinger, P., Witaszek, J., Zdanowicz, M.: Global Frobenius liftability II: Surfaces and Fano threefolds.arXiv:2102.02788(2021)

[2] Amerik, E.: Maps onto certain Fano threefolds. Doc. Math.2, 195-211 (1997) Zbl 0922.14007 MR 1467127 · Zbl 0922.14007

[3] Amerik, E., Rovinsky, M., Van de Ven, A.: A boundedness theorem for morphisms between threefolds. Ann. Inst. Fourier (Grenoble)49, 405-415 (1999) Zbl 0923.14008 MR 1697369 · Zbl 0923.14008

[4] Ash, A., Mumford, D., Rapoport, M., Tai, Y.-S.: Smooth Compactifications of Locally Symmetric Varieties. 2nd ed., Cambridge Math. Library, Cambridge Univ. Press, Cambridge (2010)Zbl 1209.14001 MR 2590897 · Zbl 1209.14001

[5] Beauville, A.: Endomorphisms of hypersurfaces and other manifolds. Int. Math. Res. Notices2001, 53-58Zbl 0986.14022 MR 1809497 · Zbl 0986.14022

[6] Borcherds, R. E., Katzarkov, L., Pantev, T., Shepherd-Barron, N. I.: Families ofK3 surfaces. J. Algebraic Geom.7, 183-193 (1998)Zbl 0946.14021 MR 1620702 · Zbl 0946.14021

[7] Borel, A., Remmert, R.: Über kompakte homogene Kählersche Mannigfaltigkeiten. Math. Ann.145, 429-439 (1961/1962)Zbl 0111.18001 MR 0145557 · Zbl 0111.18001

[8] Borger, J.: Lambda-rings and the field with one element.arXiv:0906.3146(2009)

[9] Brion, M., Kumar, S.: Frobenius Splitting Methods in Geometry and Representation Theory. Progr. Math. 231, Birkhäuser Boston, Boston, MA (2005)Zbl 1072.14066 MR 2107324 · Zbl 1072.14066

[10] Buch, A., Thomsen, J. F., Lauritzen, N., Mehta, V.: The Frobenius morphism on a toric variety. Tohoku Math. J. (2)49, 355-366 (1997)Zbl 0899.14026 MR 1464183 · Zbl 0899.14026

[11] Campana, F., Peternell, T.: Projective manifolds whose tangent bundles are numerically effective. Math. Ann.289, 169-187 (1991)Zbl 0729.14032 MR 1087244 · Zbl 0729.14032

[12] Chatzistamatiou, A., Rülling, K.: Vanishing of the higher direct images of the structure sheaf. Compos. Math.151, 2131-2144 (2015)Zbl 1330.14024 MR 3427575 · Zbl 1330.14024

[13] Debarre, O.: Images lisses d'une variété abélienne simple. C. R. Acad. Sci. Paris Sér. I Math.309, 119-122 (1989)Zbl 0699.14050 MR 1004953 · Zbl 0699.14050

[14] Deligne, P., Illusie, L.: Relèvements modulop2et décomposition du complexe de de Rham. Invent. Math.89, 247-270 (1987)Zbl 0632.14017 MR 894379 · Zbl 0632.14017

[15] Demailly, J.-P., Hwang, J.-M., Peternell, T.: Compact manifolds covered by a torus. J. Geom. Anal.18, 324-340 (2008)Zbl 144.14035 MR 2393263 · Zbl 1144.14035

[16] Demazure, M.: Automorphismes et déformations des variétés de Borel. Invent. Math. 39, 179-186 (1977)Zbl 0406.14030 MR 0435092 · Zbl 0406.14030

[17] Ejiri, S.: When is the Albanese morphism an algebraic fiber space in positive characteristic? Manuscripta Math.160, 239-264 (2019)Zbl 1444.14032 MR 3983395 · Zbl 1444.14032

[18] Esnault, H., Viehweg, E.: Lectures on Vanishing Theorems. DMV Seminar 20, Birkhäuser, Basel (1992)Zbl 0779.14003 MR 1193913 · Zbl 0779.14003

[19] Fakhruddin, N.: Questions on self maps of algebraic varieties. J. Ramanujan Math. Soc.18, 109-122 (2003)Zbl 1053.14025 MR 1995861 · Zbl 1053.14025

[20] Fulton, W.: Introduction to Toric Varieties. Ann. of Math. Stud. 131, Princeton Univ. Press, Princeton, NJ (1993)Zbl 0813.14039 MR 1234037 · Zbl 0813.14039

[21] Fulton, W.: Intersection Theory. 2nd ed., Ergeb. Math. Grenzgeb. (3) 2, Springer, Berlin (1998)Zbl 0885.14002 MR 1644323 · Zbl 0885.14002

[22] Gabber, O., Ramero, L.: Almost Ring Theory. Lecture Notes in Math. 1800, Springer, Berlin (2003)Zbl 1045.13002 MR 2004652 · Zbl 1045.13002

[23] Greb, D., Kebekus, S., Peternell, T.: Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties. Duke Math. J.165, 1965-2004 (2016)Zbl 1360.14094 MR 3522654 · Zbl 1360.14094

[24] Hartshorne, R.: Local Cohomology. A seminar given by A. Grothendieck, Harvard University, Fall 1961, Springer, Berlin (1967)Zbl 0185.49202 MR 0224620

[25] Hartshorne, R.: Algebraic Geometry. Springer, New York (1977)Zbl 0367.14001 MR 0463157 · Zbl 0367.14001

[26] Hartshorne, R.: Deformation Theory. Grad. Texts in Math. 257, Springer, New York (2010)Zbl 1186.14004 MR 2583634

[27] Hassett, B., Kovács, S. J.: Reflexive pull-backs and base extension. J. Algebraic Geom.13, 233-247 (2004)Zbl 1081.14017 MR 2047697 · Zbl 1081.14017

[28] Humphreys, J. E.: Introduction to Lie Algebras and Representation Theory. Grad. Texts in Math. 9, Springer, New York (1978)Zbl 0447.17001 MR 499562 · Zbl 0447.17001

[29] Hwang, J.-M., Mok, N.: Projective manifolds dominated by abelian varieties. Math. Z.238, 89-100 (2001)Zbl 1076.14021 MR 1860736 · Zbl 1076.14021

[30] Hwang, J.-M., Mok, N.: Birationality of the tangent map for minimal rational curves. Asian J. Math.8, 51-63 (2004)Zbl 1072.14015 MR 2128297 · Zbl 1072.14015

[31] Illusie, L.: Complexe cotangent et déformations. I. Lecture Notes in Math. 239, Springer, Berlin (1971)Zbl 0224.13014 MR 0491680 · Zbl 0224.13014

[32] Illusie, L.: Complexe de de Rham-Witt et cohomologie cristalline. Ann. Sci. École Norm. Sup. (4)12, 501-661 (1979)Zbl 0436.14007 MR 565469 · Zbl 0436.14007

[33] Jaczewski, K.: Generalized Euler sequence and toric varieties. In: Classification of Algebraic Varieties (L'Aquila, 1992), Contemp. Math. 162, Amer. Math. Soc., Providence, RI, 227-247 (1994)Zbl 0837.14042 MR 1272701 · Zbl 0837.14042

[34] Kato, F.: Log smooth deformation theory. Tohoku Math. J. (2)48, 317-354 (1996) Zbl 0876.14007 MR 1404507 · Zbl 0876.14007

[35] Kato, K.: Logarithmic structures of Fontaine-Illusie. In: Algebraic Analysis, Geometry, and Number Theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 191-224 (1989)Zbl 0776.14004 MR 1463703 · Zbl 0776.14004

[36] Katz, N. M.: Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. Inst. Hautes Études Sci. Publ. Math.39, 175-232 (1970) Zbl 0221.14007 MR 0291177 · Zbl 0221.14007

[37] Kebekus, S.: Families of singular rational curves. J. Algebraic Geom.11, 245-256 (2002)Zbl 1054.14035 MR 1874114 · Zbl 1054.14035

[38] K˛edzierski, O., Wi´sniewski, J. a. A.: Differentials of Cox rings: Jaczewski's theorem revisited. J. Math. Soc. Japan67, 595-608 (2015)Zbl 1336.14035 MR 3340188

[39] Kleiman, S. L.: The Picard scheme. In: Fundamental Algebraic Geometry, Math. Surveys Monogr. 123, Amer. Math. Soc., Providence, RI, 235-321 (2005)MR 2223410

[40] Kollár, J.: Rational Curves on Algebraic Varieties. Ergeb. Math. Grenzgeb. (3) 32, Springer, Berlin (1996)Zbl 0877.14012 MR 1440180 · Zbl 0877.14012

[41] Lange, H., Stühler, U.: Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe. Math. Z.156, 73-83 (1977)Zbl 0349.14018 MR 0472827 · Zbl 0349.14018

[42] Lauritzen, N., Mehta, V.: Fibre products of homogeneous spaces. J. Algebra189, 568-576 (1997)Zbl 0894.14023 MR 1438189 · Zbl 0894.14023

[43] Lazarsfeld, R.: Some applications of the theory of positive vector bundles. In: Complete Intersections (Acireale, 1983), Lecture Notes in Math. 1092, Springer, Berlin, 29-61 (1984)Zbl 0547.14009 MR 775876

[44] Liedtke, C., Satriano, M.: On the birational nature of lifting. Adv. Math.254, 118-137 (2014)Zbl 1319.14015 MR 3161094 · Zbl 1319.14015

[45] Mehta, V. B., Nori, M. V.: Semistable sheaves on homogeneous spaces and abelian varieties. Proc. Indian Acad. Sci. Math. Sci.93, 1-12 (1984)Zbl 0592.14017 MR 796768 · Zbl 0592.14017

[46] Mehta, V. B., Srinivas, V.: Varieties in positive characteristic with trivial tangent bundle. Compos. Math.64, 191-212 (1987)Zbl 0639.14024 MR 916481 · Zbl 0639.14024

[47] Nakayama, N., Zhang, D.-Q.: Polarized endomorphisms of complex normal varieties. Math. Ann.346, 991-1018 (2010)Zbl 1189.14043 MR 2587100 · Zbl 1189.14043

[48] Occhetta, G., Wi´sniewski, J. A.: On Euler-Jaczewski sequence and Remmert-van de Ven problem for toric varieties. Math. Z.241, 35-44 (2002)Zbl 1022.14016 MR 1930984 · Zbl 1022.14016

[49] Paranjape, K. H., Srinivas, V.: Self-maps of homogeneous spaces. Invent. Math.98, 425-444 (1989)Zbl 0697.14037 MR 1016272 · Zbl 0697.14037

[50] Remmert, R., van de Ven, T.: Über holomorphe Abbildungen projektiv-algebraischer Mannigfaltigkeiten auf komplexe Räume. Math. Ann.142, 453-486 (1960/1961) Zbl 0099.16403 MR 0145556 · Zbl 0099.16403

[51] Sancho de Salas, C.: Complete homogeneous varieties: structure and classification. Trans. Amer. Math. Soc.355, 3651-3667 (2003)Zbl 1087.14036 MR 1990167 · Zbl 1087.14036

[52] Serre, J.-P.: Groupes algébriques et corps de classes. Hermann, Paris (1959) Zbl 0097.35604 MR 0103191 · Zbl 0097.35604

[53] Snow, D. M.: Cohomology of twisted holomorphic forms on Grassmann manifolds and quadric hypersurfaces. Math. Ann.276, 159-176 (1986)Zbl 0596.32016 MR 863714 · Zbl 0596.32016

[54] Snow, D. M.: Vanishing theorems on compact Hermitian symmetric spaces. Math. Z. 198, 1-20 (1988)Zbl 0631.32025 MR 938025 · Zbl 0631.32025

[55] Stacks Project Authors: Stacks Project.http://stacks.math.columbia.edu(2014)

[56] Totaro, B.: Bott vanishing for algebraic surfaces. Trans. Amer. Math. Soc.373, 3609- 3626 (2020)Zbl 1437.14028 MR 4082249 · Zbl 1437.14028

[57] van der Geer, G., Katsura, T.: On the height of Calabi-Yau varieties in positive characteristic. Doc. Math.8, 97-113 (2003)Zbl 1074.14524 MR 2029163 · Zbl 1074.14524

[58] Winkelmann, J.: On manifolds with trivial logarithmic tangent bundle. Osaka J. Math. 41, 473-484 (2004)Zbl 1058.32011 MR 2069097 · Zbl 1058.32011

[59] Xin, H.: OnW2-lifting of Frobenius of algebraic surfaces. Collect. Math.67, 69-83 (2016)Zbl 1386.14092 MR 3439841 · Zbl 1386.14092

[60] Zdanowicz, M.: Liftability of singularities and their Frobenius morphism modulop2. Int. Math. Res. Notices2018, 4513-4577Zbl 1423.14168 MR 3830576 · Zbl 1423.14168

[61] Zhang, D.-Q.: Polarized endomorphisms of uniruled varieties. Compos. Math.146, 145-168 (2010)Zbl 1198 · Zbl 1198.14019