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Stability of valuations and Kollár components. (English) Zbl 1471.14076
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The authors investigate klt singularities using tools from the theory of K-stability. This is natural because klt singularities are a local analog of Fano varieties. Let $o \in (X, D)$ be a klt singularity, then a proper birational morphism $\mu : Y \rightarrow X$ provides a Kollár component S if μ is an isomorphism over $X \setminus \{o\}$, $\mu^{-1}(o) = S$ is a \mathbb{Q} -Cartier prime divisor such that $(Y, S + \mu^{-1}D)$ is plt and $-S$ is μ -ample. Such components always exist and have the structure of a (log-)Fano variety. The authors prove that amongst all Kollár components of a klt singularity $o \in (X, D)$ there is at most one such component that is (log-)K-semistable. This component corresponds to the unique divisorial valuation that minimizes the normalized volume function introduced by Chi Li.

Reviewer: [Christopher Hacon \(Salt Lake City\)](#)

MSC:

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13A18 Valuations and their generalizations for commutative rings

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