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Principalization of ideals on toroidal orbifolds. (English) Zbl 1473.14026
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A theory of the resolution of singularities should not only prove that every space Z can be resolved by a modification $Z' \rightarrow Z$, but the resolution should be

- canonical in the sense that the theory distinguishes one resolution $Z_{res} \rightarrow Z$;
- constructive in the sense that there is an explicit algorithmic procedure to obtain the distinguished resolution $Z_{res} \rightarrow Z$;
- functorial in the sense that for every smooth morphism $Y \rightarrow Z$, the resolution $Y_{res} \rightarrow Y$ is the pull-back (in the appropriate category) of $Z_{res} \rightarrow Z$.

The article is the first in an announced series which establishes such a resolution for morphisms $Z \rightarrow B$ of fine saturated logarithmic Deligne-Mumford stacks. The article itself deals only with the case $B = \text{Spec}(k)$ the trivial log point, i.e., with the absolute case.

The starting point is resolution of logarithmic schemes. The authors ask for an extended functoriality principle, i.e., the resolution should not only be functorial for classically smooth maps $Y \rightarrow Z$ but also for log smooth maps $Y \rightarrow Z$. This makes it necessary to resolve via blowups of Kummer centers, i.e., ideals in the Kummer étale topology of Z . Such a blowup is not necessarily a logarithmic scheme but a logarithmic Deligne-Mumford stack; thus the natural setup for the resolution are (fine saturated) logarithmic Deligne-Mumford stacks. A log smooth Deligne-Mumford stack is called a toroidal orbifold.

The algorithm comes in two versions, an embedded one – the principalization of ideals on a toroidal orbifold – and a non-embedded one – the resolution of singularities.

The algorithm does not specialize to a classical one for trivial logarithmic structures. When it starts with a variety Z with trivial logarithmic structure, then Z_{res} , in general, neither has trivial logarithmic structure nor is a scheme but a honest toroidal orbifold.

The algorithm is less complicated than algorithms for resolving singularities of classical schemes since it does not need to take separate care about the exceptional divisor, but it is nonetheless intricate. It employs a logarithmic version of the induction over hypersurfaces of maximal contact to reduce the log order of marked ideals. Additionally, in several so-called cleaning processes it must be ensured that the ideal to resolve has a nice interplay with the logarithmic structure.

After this article was finished, Ming Hao Quek – a student of one of the authors – has found a simpler, more canonical, and presumably faster algorithm to resolve logarithmic singularities in the same setup, see [M. H. Quek, “Logarithmic resolution via weighted toroidal blow-ups”, Preprint, [arXiv:2005.05939](https://arxiv.org/abs/2005.05939)]. This is achieved by allowing more general centers which are no longer ideals in the Kummer étale topology; it is a logarithmic variant of the *dream algorithm* of [D. Abramovich et al., “Functorial embedded resolution via weighted blowings up”, Preprint, [arXiv:1906.07106](https://arxiv.org/abs/1906.07106)].

Reviewer: [Simon Felten \(Mainz\)](#)

MSC:

[14E15](#) Global theory and resolution of singularities (algebraic-geometric aspects)
[14A20](#) Generalizations (algebraic spaces, stacks)
[14A21](#) Logarithmic algebraic geometry, log schemes

Cited in **5** Documents

Keywords:

[resolution of singularities](#); [logarithmic geometry](#); [algebraic stacks](#)

Full Text: [DOI](#) [arXiv](#)

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