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Uniform K-stability and asymptotics of energy functionals in Kähler geometry. (English)

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Summary: Consider a polarized complex manifold (X, L) and a ray of positive metrics on L defined by a positive metric on a test configuration for (X, L) . For many common functionals in Kähler geometry, we prove that the slope at infinity along the ray is given by evaluating the non-Archimedean version of the functional (as defined in our earlier paper [Ann. Inst. Fourier 67, No. 2, 743–841 (2017; Zbl 1391.14090)]) at the non-Archimedean metric on L defined by the test configuration. Using this asymptotic result, we show that coercivity of the Mabuchi functional implies uniform K-stability, as defined in [R. Dervan, Int. Math. Res. Not. 2016, No. 15, 4728–4783 (2016; Zbl 1405.32032); the authors, loc. cit.]. As a partial converse, we show that uniform K-stability implies coercivity of the Mabuchi functional when restricted to Bergman metrics.

MSC:

53C55 Global differential geometry of Hermitian and Kählerian manifolds
14L24 Geometric invariant theory
32P05 Non-Archimedean analysis
32Q20 Kähler-Einstein manifolds
32Q26 Notions of stability for complex manifolds

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Keywords:

K-stability; Kähler geometry; canonical metrics; non-Archimedean geometry

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