Negreanu, M.; Tello, J. I.; Vargas, A. M.
On a fully parabolic chemotaxis system with nonlocal growth term. (English) Zbl 1479.35521

In this article, the authors studied the following parabolic-parabolic chemotaxis system with logistic growth term, posed in a bounded domain $\Omega \subset \mathbb{R}^N$:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - \nabla \cdot (\chi u^m \nabla v) + f(u), \\
\frac{\partial v}{\partial t} &= \Delta v - v + u^\gamma, \text{ in } \Omega \times (0, T),
\end{align*}
\]

supplemented with homogeneous Neumann boundary condition $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on $\partial \Omega$, where $\chi > 0$, $m > 1$ and the nonlinearity $f(u)$ is given by

\[f(u) = u(a_0 - a_1 u^\alpha + a_2 \int_\Omega u^\alpha \, dx), a_0, a_1 > 0, \alpha \geq 1.\]

The biological meaning of the above PDE is as follows: Here $u$ is the population’s density of a biological species, $v$ is the density of the chemical substance responsible for the chemotactic process. The individuals of the species are able to recognize the chemical signal and measure its concentration. They tend to move towards the higher concentrations of the chemical if they like it ($\chi > 0$) or, move away from it if they dislike it ($\chi < 0$). The authors consider the sensitivity parameter $\chi > 0$. The parameter $m > 1$ describes that there is a reinforcement in the direction of $\nabla v$ where the population density $u > 1$ and weaker if $u < 1$, and the term $\Delta u$ represents the natural diffusion of the densities. The movement of the cell density is aided by a nonlinear interaction (represented by $f(u)$) among the cells. Here the first two terms in $f(u)$ describe the local competition: $a_0 > 0$ induces an exponential growth for the species, while $-a_1 u^\alpha$ with $a_1 > 0$ limits the growth, and the last term $a_2 \int_\Omega u^\alpha$ describes the global interaction. If $a_2 < 0$, there is a non-local competition among the individuals, and if $a_2 > 0$ the individuals cooperate globally for the survival of the species.

The authors studied the system (1) corresponding to the homogeneous Neumann boundary condition and non-negative initial data $u_0, v_0$ satisfying

\[u_0, v_0 \in W^{2,q}(\Omega) \text{ for all } q < \infty \text{ and } \frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0 \text{ on } \partial \Omega.\]  

Under the following assumptions on the parameters

\[\alpha \geq 1, \ m > 1, \ \gamma \geq 1, \ \alpha + 1 > m + \gamma, \ a_1 - a_2 |\Omega| > 0, \ a_1 > 0, \ a_0 > 0, \]

the authors studied the global existence and asymptotic behaviour of the solutions. The results obtained in this article are as follows:

**Global existence:** Under the assumptions (1) and (2) there exists a unique global solution to (3) satisfying

\[\sup_{t > 0} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty.\]

Moreover, the condition $\gamma \geq 1$ can be relaxed to $\gamma > 0$.

**Asymptotic behaviour:** If in addition, $\min\{a_0, a_2\} > 0$ is assumed, then the solution converges to the unique positive steady state $(u^*, (u^*)^\gamma)$ in $L^p$ for all $p \in [1, \infty]$ as $t \to \infty$:

- $||u(\cdot, t) - u^*||_{L^p(\Omega)} + ||v(\cdot, t) - (u^*)^\gamma||_{L^p(\Omega)} \to 0$, as $t \to \infty,$

where

\[u^* = \frac{a_0^\frac{1}{\alpha}}{a_1 - a_2 |\Omega|^{\frac{1}{\gamma}}}.\]

On the other hand
If $a_1 > 0$, $a_0 < 0$ and $a_1 - a_2 |\Omega| \geq 0$, then the solution (if exists globally) converges to the zero steady state with an exponential rate. In precise, the following estimate holds:

$$||u||_{L^p(\Omega)} \leq c_1 e^{-c_2 t},$$

where $c_1, c_2 > 0$, $p = \frac{2N}{N(2 - \beta) - 2\beta}$, $\beta \in [0, 1]$. 

The proof of global existence is achieved by first obtaining local existence and then by getting appropriate estimates on the local solutions. On the other hand, the proof of long time asymptotic of the solutions requires a lower bound of the total mass $\int_{\Omega} u$ for all time, which is done by tracking the evolution of $\int_{\Omega} u^{-\beta}$ for some $\beta > 0$ in spirit of the related works [M. Mizukami and T. Yokota, J. Differ. Equations 261, No. 5, 2650–2669 (2016; Zbl 1339.35197); M. Negreanu and J. I. Tello, J. Differ. Equations 258, No. 5, 1592–1617 (2015; Zbl 1310.35040); K. Fujie et al., Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods 109, 56–71 (2014; Zbl 1297.35051); Y. Tao and M. Winkler, J. Differ. Equations 259, No. 11, 6142–6161 (2015; Zbl 1321.35084)].

Moreover, the authors obtained various numerical results demonstrating that the assumptions on the parameters (3) play an essential role in respect of the global existence and asymptotic behaviour of the solutions:

- If the assumption $\alpha + 1 > m + \gamma$ or $a_1 > a_2 |\Omega|$ is violated, then the $(|| \cdot ||_{L^\infty}$ norm of the) discrete solution blows-up in finite time.
- If $a_0 < 0$ is assumed, then the discrete solution converges to the steady state $(0, 0)$.

Reviewer: Debabrata Karmakar (Bangalore)

MSC:

35K51 Initial-boundary value problems for second-order parabolic systems
35B40 Asymptotic behavior of solutions to PDEs
92C17 Cell movement (chemotaxis, etc.)
35B35 Stability in context of PDEs
35K59 Quasilinear parabolic equations

Keywords:
chemotaxis; global existence of solutions; asymptotic behaviour; nonlocal growth terms

Full Text: DOI

References:


This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.