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Nondegenerate spheres in four dimensions. (English) Zbl 1495.52019

Summary: Non-degeneracy was first defined for hyperplanes by Elekes-Tóth, and later extended to spheres by Apfelbaum-Sharir: given a set $P$ of $m$ points in $\mathbb{R}^d$ and some $\beta \in (0, 1)$, a $(d-1)$-dimensional sphere (or a $(d-1)$-sphere) $S$ in $\mathbb{R}^d$ is called $\beta$-nondegenerate with respect to $P$ if $S$ does not contain a proper subsphere $S'$ such that $|S' \cap P| \geq \beta |S \cap P|$. Apfelbaum-Sharir found an upper bound for the number of incidences between points and nondegenerate spheres in $\mathbb{R}^3$, which was recently used by Zahl to obtain the best known bound for the unit distance problem in three dimensions. In this paper, we show that the number of incidences between $m$ points and $n\beta$-nondegenerate 3-spheres in $\mathbb{R}^4$ is $O_{\beta, \varepsilon}(n^{15/19+\varepsilon} m^{16/19} + mn^{2/3})$. As a consequence, we obtain a bound of $O_\varepsilon(n^{2+4/11+\varepsilon})$ on the number of similar triangles formed by $n$ points in $\mathbb{R}^3$, an improvement over the previously best known bound $O(n^{2+2/3})$. While proving this, we find it convenient to work with a more general definition of nondegeneracy: a bipartite graph $G = (P, Q)$ is called $\beta$-nondegenerate if $|N(q_1) \cap N(q_2)| < \beta |N(q_1)|$ for any two distinct vertices $q_1, q_2 \in Q$; here $N(q)$ denotes the set of neighbors of $q$ and $\beta$ is some positive constant less than 1. A $\beta$-nondegenerate graph can have up to $\Theta(|P||Q|)$ edges without any restriction, but must have much fewer edges if the graph is semi-algebraic or has bounded VC-dimension. We show that Elekes-Tóth’s bound for nondegenerate hyperplanes, Apfelbaum-Sharir’s bound for nondegenerate spheres in $\mathbb{R}^3$, and our new bound for nondegenerate spheres in $\mathbb{R}^4$, all hold under this new definition.

MSC:

52C35 Arrangements of points, flats, hyperplanes (aspects of discrete geometry)
05C20 Directed graphs (digraphs), tournaments

Keywords:
incidence geometry; non-degenerate spheres; similar simplices

Full Text: DOI arXiv

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