

Statha, Marina

Ricci flow on certain homogeneous spaces. (English) Zbl 1502.53078
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The author studies the normalized Ricci flow of invariant metrics on certain homogeneous spaces with three and four isotropy summands, such as generalized Wallach spaces, certain Stiefel manifolds $V_k\mathbb{R}^n$ and generalized flag manifolds. On such spaces, the normalized Ricci flow is equivalent to a homogeneous system of differential equations in \mathbb{R}^3 and \mathbb{R}^4 . So in order to study the behavior of such systems at infinity, a method introduced by Poincaré will be used, the so-called Poincaré compactification. The important point is the fact that if one knows the behavior of a projected vector field around the equator in the compactification, then one knows the behavior of the vector field in the neighborhood of infinity.

The main contribution of the present work is that by using the Poincaré compactification, one confirms previously obtained Einstein metrics as fixed points of dynamical systems deduced from normalized Ricci flow, but one also detects new homogeneous Einstein metrics on certain spaces. It is proved that for a generalized Wallach space, some Stiefel manifolds, and for some flag manifolds, the normalized Ricci flow of an invariant Riemannian metric has a finite number of singularities at infinity. Specific situations are analysed in order to give the exact number of singularities. And these fixed points correspond (up to scale) to the G -invariant Einstein metrics on the corresponding homogeneous space.

Reviewer: [Gabriela Paola Ovando \(Rosario\)](#)

MSC:

- 53C25 Special Riemannian manifolds (Einstein, Sasakian, etc.)
- 53C30 Differential geometry of homogeneous manifolds
- 53E20 Ricci flows
- 34A26 Geometric methods in ordinary differential equations

Keywords:

[Ricci flow](#); [Einstein metric](#); [Poincaré compactification](#); [generalized Wallach space](#); [Stiefel manifold](#); [generalized flag manifold](#); [Gröbner basis](#)

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